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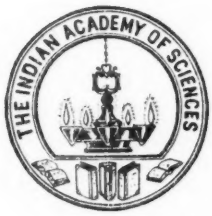

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ON A PROBLEM RELATING TO A TETRAHEDRON.

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1. In two papers published in the previous number of these *Proceedings*, we solved the problem of determining the tetrahedron of maximum volume, the areas of whose faces are given. It was proved that the maximum is attained when the tetrahedron is orthogonal, and that—as one of us (K. S. K. Iyengar) succeeded in proving—in the particular case of three dimensions there is only one such tetrahedron. The analogous problem in any number of dimensions was also solved partially. By an interesting transformation of the same problem we now prove the above results of our earlier papers in a far more elegant way and we prove further that in the case of n dimensions also there is only one such orthogonal simplex, with assigned values for the $(n-1)$-dimensional volumes of its face-simplexes. Finally, we solve by these methods an isoperimetrical problem of Steinitz.

2. Let $OP_1P_2 \cdots P_n$ be a simplex in n dimensions and $a_1 \cdots a_n$, a_{n+1} the $(n-1)$-dimensional volumes of the face-simplexes $OP_2 \cdots P_n$, $OP_1P_3 \cdots P_n$, $OP_1P_2 \cdots P_{n-1}$, $P_1P_2 \cdots P_n$. The problem is to determine the extrema for the volume of the simplex when the a_r's are all given. We first prove that a real simplex exists if and only if the sum of any n of the a_r's is greater than the remaining one. We also prove that the lower bound of the volume is zero and that the maximum is attained when the simplex is orthogonal, which is uniquely determined when the a_r's are given. For proving these results the following transformation into another which is also an isoperimetric one is fundamental. The artifice employed is to obtain another simplex, the $(n+1)$ sides of which, forming a one-dimensional closed chain, are of lengths $a_1, a_2 \cdots a_{n+1}$ and whose volume is a multiple of the $(n-1)$th power of the volume of the original simplex. It will be seen that by this means the solution of our problem becomes almost immediate. Let $\overrightarrow{OP_1}, \overrightarrow{OP_2}, \cdots \overrightarrow{OP_n}$ be represented by the vectors $\vec{A}_1, \vec{A}_2, \cdots \vec{A}_n$ and let the $(n-1)$-dimensional volumes of the faces

$OP_2P_3 \cdots P_n$, $OP_1P_3 \cdots P_n$, $\cdots$ $P_1P_2 \cdots P_n$ of the simplex $OP_1P_2 \cdots P_n$ be a_1, $a_2 \cdots a_n$, a_{n+1} respectively. If we represent by $\vec{A}_1'$ the vector product $(\vec{A}_2\vec{A}_3 \cdots \vec{A}_n)$ and similarly the other vector products, it is easy to see that

(i) $|\vec{A}_1'| = (n-1)!\, a_1, \cdots |\vec{A}_n'| = (n-1)!\, a_n$

(ii) if $-\vec{A}'_{n+1} = \vec{A}_1' + \vec{A}_2' + \cdots + \vec{A}_n'$, then the vector $\vec{A}'_{n+1}$ will be perpendicular to the face $P_1P_2 \cdots P_n$, and $|\vec{A}'_{n+1}| = (n-1)\, a_{n+1}$. If therefore we draw OT_1, $T_1T_2, \cdots T_{n-1}T_n$ to represent the vectors $\frac{1}{(n-1)!}\vec{A}_1'$, $\frac{1}{(n-1)!}\vec{A}_2', \cdots \frac{1}{(n-1)!}\vec{A}_n'$ respectively, then OT_n will represent the vector $\frac{1}{(n-1)!}\vec{A}'_{n+1}$, since $\sum_{r=1}^{n+1} \vec{A}_r' = 0$.

Hence the necessary and sufficient condition for the existence of simplexes having the given numbers $a_1, a_2, \cdots a_{n+1}$ for face-volumes is the existence of a closed linkage (polygon) $OT_1T_2 \cdots T_nO$ in space such that $|OT_1| = a_1$, $|T_1T_2| = a_2 \cdots |T_{n-1}T_n| = a_n$ and $|OT_n| = a_{n+1}$. But this means that the sum of any n of the numbers $a_1, a_2, \cdots a_{n+1}$ should be greater than the remaining one; or, if $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \leqslant a_{n+1}$ then the sole condition for the existence of simplexes with given $(n-1)$-dimensional volumes of the faces is $a_1 + a_2 + \cdots + a_n > a_{n+1}$.

3. Now the volume of the simplex $OP_1P_2 \cdots P_n$ is

$$V = \frac{1}{n!} \| \vec{A}_1\vec{A}_2 \cdots \vec{A}_n \|$$

and that of the simplex $OT_1T_2 \cdots T_n$ is

$$V = \frac{1}{n!} \left\| \frac{\vec{A}_1'}{(n-1)!}, \frac{\vec{A}_2'}{(n-1)!}, \cdots \frac{\vec{A}_n'}{(n-1)!} \right\|$$

$$\therefore \quad V = \frac{1}{n!\,\{(n-1)!\}^n} \left\| \vec{A}_1' \vec{A}_2' \cdots \vec{A}_n' \right\|$$

$$= \frac{1}{n!\,\{(n-1)!\}^n} \left\| \vec{A}_1 \vec{A}_2 \cdots \vec{A}_n \right\|^{n-1}$$

$$= \frac{1}{n!\,\{(n-1)!\}^n} \left\{(n!)\, V\right\}^{n-1}$$

whence $$V = \frac{\{(n-1)!\}^2}{n^{n-2}} \overline{V},$$

so that the problem of finding the bounds of the volumes of all simplexes having given face-volumes $a_1, a_2, \cdots a_{n+1}$ such as $OP_1P_2 \cdots P_n$ is

completely equivalent to the problem of finding the bounds of the volumes of all simplexes $OT_1T_2 \cdots T_n$ such that $|OT_1| = a_1$, $(T_1T_2) = a_2 \cdots |T_{n-1}T_n| = a_n$, $|OT_n| = a_{n+1}$.

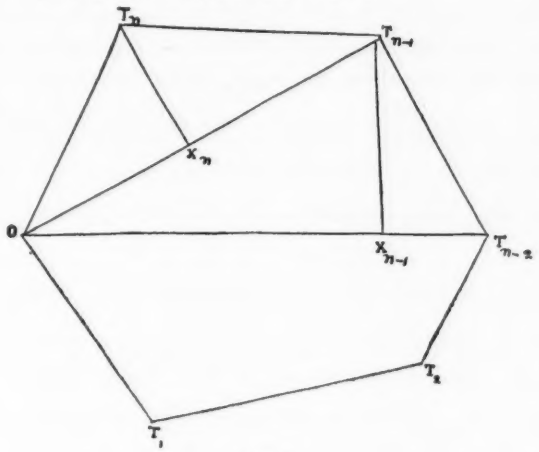

FIG. 1.

Let ϕ_n be the angle between $T_n\bar{X}_n$ and the perpendicular to the space $OT_1T_2 \cdots T_{n-1}$ in the n-space, where $T_n\bar{X}_n$ is the perpendicular to OT_{n-1} in the plane $OT_{n-1}T_n$.

Similarly let ϕ_{n-1} be the angle between $T_{n-1}\bar{X}_{n-1}$ and the perpendicular to the space $OT_1T_2 \cdots T_{n-2}$ in the $(n-1)$-space defined by $OT_1T_2 \cdots T_{n-1}$, where $T_{n-1}\bar{X}_{n-1}$ is defined similarly to $T_n\bar{X}_n$; and so on. Then the volume of the simplex is given by

$$\frac{1}{n} \cdot T_n\bar{X}_n \cdot (n-1)\text{-dimensional volume of } OT_1T_2 \cdots T_{n-1} \cdot \cos\phi_n$$

and by repeated application of this result we find for the volume the expression

$$\frac{1}{n!} \cdot T_n\bar{X}_n \cdot T_{n-1}\bar{X}_{n-1} \cdots T_2\bar{X}_2 \cdot OT_1 \cos\phi_n \cos\phi_{n-1} \cdots \cos\phi_3.$$

Let us denote OT_2 by x_1, OT_3 by $x_2 \cdots OT_{n-1}$ by x_{n-2} and the area of any triangle with sides a, b, c by the symbol (a, b, c).

Then

$$T_n\bar{X}_n = \frac{2\,(a_{n+1}, a_n, x_{n-2})}{x_{n-2}}$$

$$T_{n-1}\bar{X}_{n-1} = \frac{2\,(x_{n-2}, x_{n-3}, a_{n-1})}{x_{n-3}}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$T_2\bar{X}_2 = \frac{2\,(a_1\, a_2\, x_1)}{OT_1}.$$

The volume of the simplex $OT_1'T_2 \cdots T_n$ is therefore given by

$$\frac{2^{n-1}}{n!} \frac{(a_{n+1}, a_n, x_{n-2})\ (x_{n-2}, x_{n-3}, a_{n-1}) \cdots (a_1, a_2, x_1)}{x_1 x_2 \cdots x_{n-2}} \times \cos\phi_n \cos\phi_{n-1} \cdots \cos\phi_3.$$

We see at once that the lower bound of the volumes of all such simplexes is zero, because any one of the angles $\phi_3, \phi_4 \cdots \phi_n$ or all of them can be made as nearly equal to $\frac{\pi}{2}$ as we like. We also see at once that in the maximum simplex, all the angles $\phi_n, \phi_{n-1} \cdots \phi_3$ are zero. In other words, going back to the vector notation, since $OT_1 = \vec{A}_1'$, etc., we see that

$$[\vec{A}_1'\vec{A}_2'] \text{ and } [\vec{A}_3', \vec{A}_1' + \vec{A}_2']$$

are perpendicular to each other, the vector products being defined in the space given by $\vec{A}_1', \vec{A}_2', \vec{A}_3'$.

Instead of taking the simplex in the order $OT_1'T_2 \cdots T_n$ where $OT_1 = \vec{A}_1'$, $OT_2 = \vec{A}_2'$, etc., we could form another simplex $OT_1''T_2' \cdots T_n'$ such that $OT_1' = \vec{A}_r$, $OT_2' = \vec{A}_s$, etc., and it is easy to see that

$$\text{Volume } (OT_1' \cdots T_n') = \text{Volume } (OT_1 \cdots T_n)$$

and we get for the maximum simplex corresponding conditions. Therefore for the maximum simplex, all the conditions are given by the following:—

The vectors $[\vec{A}_r'\vec{A}_s']$, $[\vec{A}_p', \vec{A}_r' + \vec{A}_s']$ should be perpendicular to each other, the vector-products being defined in the space given by $\vec{A}_r', \vec{A}_s', \vec{A}_p'$.

In the first papers of ours published in the previous number of these *Proceedings* we showed that for the maximum simplex with given faces, opposite edges are mutually perpendicular. This follows at once from the above if we go back from the vectors $\vec{A}_r'$ to the vectors $\vec{A}_r$.

4. We shall now prove that the maximum simplex $OT_1T_2 \cdots T_n$ with given sides $OT_1 = a_1$, etc., is unique.

Sufficient amount of complexity is introduced if we restrict ourselves to 5 dimensions, the proof for n dimensions being the same as for 5.

Let the maximum simplex be $OT_1T_2 \cdots T_5$ where $OT_1 = a_1$, $OT_2 = a_5$, $\cdots$ $OT_5 = a_6$.

Let $OT_2 = x$, $OT_3 = y$, $OT_4 = z$. Then the volume of the simplex is

$$V = K \cdot \frac{(a_1 a_2 x)\ (x a_3 y)\ (y a_4 z)\ (z a_5 a_6)}{xyz},$$

where K is a constant.

$$V^2 = K^2 \cdot \frac{\{(a_1 + a_2)^2 - x^2\}\{x^2 - (a_1 - a_2)^2\} \cdots \{\quad\}}{x^2 y^2 z^2}.$$

Put $x^2 = u$, $y^2 = v$, $z^2 = w$ and designate $(a_1 a_2 x)^2$ by $[a_1 a_2 u]$, etc. We shall prove that $\log V$ is a convex function of (u, v, w).

Now $2 \log V = \text{constant} + \log [a_1, a_2, u] + \log \{[u a_3 v]/u\} + \log \{[v a_4 w]/v\}$
$+ \log \{[w a_5 a_6]/w\}$.

It is easy to prove that if $f(u, v, w)$ is convex so is also $\log f(u, v, w)$. We now prove that $[a_1 a_2 u]$ is convex ; for

$[a_1 a_2 u] = K\{(a_1 + a_2)^2 - u\}\{u - (a_1 - a_2)^2\}$, $[K = \frac{1}{16}]$ and so $\frac{d^2}{du^2}[a_1, a_2, u] < 0$.

Similarly $[w a_5 a_6]/w = K w^{-1}\{(a_5 + a_6)^2 - w\}\{w - (a_5 - a_6)^2\}$

$$= A + Bw - \frac{K(a_5 + a_6)^2 (a_5 - a_6)^2}{w}$$

$$\therefore \frac{d^2}{dw^2}\left\{[w a_5 a_6]\, w^{-1}\right\} < 0.$$

Consider now $f = u^{-1}[u a_3 v]$

$$= \frac{K\{u - (a_3 - \sqrt{v})^2\}\{(a_3 + \sqrt{v})^2 - u\}}{u}$$

$$= \frac{[-u^2 + u\{(a_3 - \sqrt{v})^2 + (a_3 + \sqrt{v})^2\} - (a_3^2 - v)^2]\, K}{u}$$

$$= \left[-u + 2(a_3^2 + v) - \frac{(a_3^2 - v)^2}{u}\right] K$$

$$\therefore \frac{1}{K}\frac{\partial^2 f}{\partial u^2} = -\frac{2(a_3^2 - v)^2}{u^3}; \quad \frac{1}{K}\frac{\partial^2 f}{\partial v^2} = -\frac{2}{u}; \quad \frac{1}{K}\frac{\partial^2 f}{\partial u \partial v} = -\frac{2(a_3^2 - v)}{u^2}$$

so that

$$\frac{\partial^2 f}{\partial u^2}\frac{\partial^2 f}{\partial v^2} = \left(\frac{\partial^2 f}{\partial u \partial v}\right)^2.$$

The quadratic form formed by the second partial differential coefficients of the logarithm of each of the functions is therefore negative definite and hence $\log V$ is a convex function of u, v, w. Now in the maximum simplex $\frac{\partial \overline{V}}{\partial u} = \frac{\partial \overline{V}}{\partial v} = \frac{\partial \overline{V}}{\partial w} = 0$ (the angles ϕ_5, ϕ_4, ϕ_3 being zero as before). The range of admissible values of u, v, w forms a continuum K. If there are two maxima in the continuum at P_1 and P_2 we can join $P_1 P_2$ by a curve L lying in K and in that curve there must be a minimum of $\log V$ for variations along L. But that is impossible since the quadratic form of the second order differential coefficients of $\log V$ is negative definite.

This therefore proves that the maximum simplex is unique. Going back from the simplex $OT_1T_2 \cdots T_n$ to the simplex $OP_1P_2 \cdots P_n$ we can state the theorem as follows :—

(1) All the stationary simplexes $OP_1P_2 \cdots P_n$ having given volumes [$(n-1)$-dimensional] for its faces are orthogonal (*vide* also **K. S. K.** Iyengar's paper on this in the preceding number of the *Proceedings*).

(2) There is only one such orthogonal simplex which is the maximum.

5. We will now shew that, by this method of ours, we can prove a result of Steinitz that if the sum of the four areas of a tetrahedron be given then the tetrahedron has maximum volume when all the faces are equal and it is regular (*incidentally orthogonal*). This result could be easily generalised to n dimensions. As the method is the same for all dimensions we will give here a proof for 3 dimensions.

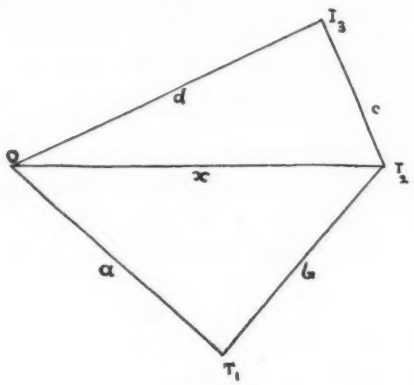

FIG. 2.

Adopting our old notation, if the volume of the tetrahedron $OP_1P_2P_3$ with given areas a, b, c, d for its faces be V, then $V^2 = \frac{4}{3}\bar{V}$,
where $\bar{V}$ = volume of $(OT_1T_2T_3)$ in our usual notation.
Let $\bar{V}$ be the volume of the maximum simplex with sides a, b, c, d.
Then

$$\bar{V} = K.\ \frac{(\text{area } OT_1T_2)\cdot(\text{area } OT_2T_3)}{OT_2}$$

$$= K.\ \frac{(a, b, x)\,(c, d, x)}{x}.$$

[Note that x is a function of a, b, c, d and that the angle between the planes OT_1T_2, OT_2T_3 is $\frac{\pi}{2}$.]

Let us now deform the triangle $OT_1'T_2$ to another $OT_1''T_2$ such that

$$OT_1' = T_1''T_2 \text{ and } OT_1' + T_1''T_2 = a + b,$$

(T_1' being in the same plane OTT_2).

Then the volume of the simplex $OT_1''T_2T_3$

$$= K. \frac{\left(\frac{a+b}{2}, \frac{a+b}{2}, x\right)(c, d, x)}{x}$$

$$\geqslant \text{Volume } OT_1'T_2'T_3$$

since

$$\left(\frac{a+b}{2}, \frac{a+b}{2}, x\right) \geqslant (a, b, x).$$

It therefore follows at once that of all the tetrahedra $OT_1'T_2'T_3$ such that $OT_1 + T_1'T_2 + T_2'T_3 + T_3O = 4a$, the one for which

(i) $OT_1 = T_1'T_2 = \cdots = a$,

(ii) the angles between the planes $(OT_1'T_2, OT_3'T_2)$ and also the angles between the planes $(OT_1'T_3, T_1'T_2'T_3)$ are right angles

is the maximum. Going back to the original tetrahedron $OP_1P_2P_3$ the maximum tetrahedron is readily seen to be a regular one.

REACTIVITY OF THE DOUBLE BOND IN COUMARINS AND RELATED $\alpha:\beta$-UNSATURATED CARBONYL COMPOUNDS.

Part VI. Action of Mercuric Acetate on the Methyl Ethers of Coumarinic and Coumaric Acids.

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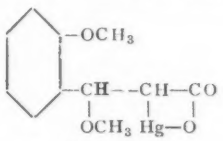
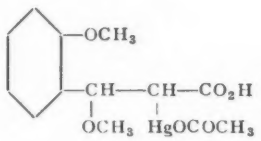
Received April 23, 1938.

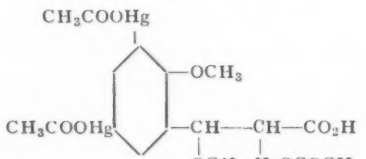
In continuation of our work described in Part V[1] the action of mercuric acetate on the methyl ethers of coumarinic and coumaric acids and their 5-nitro derivatives has now been examined. Biilmann[2] found that the first two compounds underwent addition only and that the anhydride form (I) was produced. In the case of coumaric acid it has already been shown by Rao and Seshadri[3] that besides addition at the double bond, mercuration of the benzene ring also takes place and that the product has the carboxylic formula corresponding to (II).



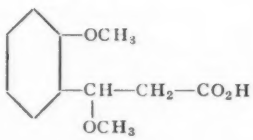
(I)



(II)



(III)



(IV)

At the temperature of the room (28°) the methyl ether of coumarinic acid has now been found to undergo simple addition at the double bond to form α-acetoxymercuri-β : 2-dimethoxydihydrocinnamic acid (II). When the addition product is dissolved in aqueous sodium hydroxide and the

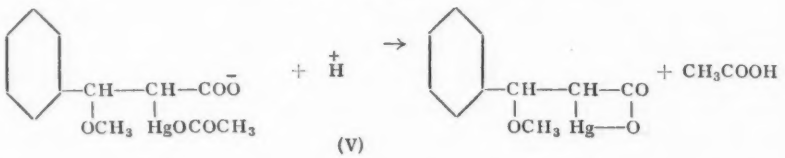
solution rendered acid with sulphuric acid the corresponding anhydro form (I) is produced. On heating, however, mercuration of the benzene ring also takes place so as to yield β : 2-dimethoxy-α : 3 : 5-triacetoxymercuridihydrocinnamic acid (III). The behaviour of this compound is very similar to that obtained from *para*-methoxycinnamic acid.[1] The methyl ether of coumaric acid is slower in reaction. In the cold it reacts slowly and no definite compound could be isolated, since besides addition there is some mercuration also. At the boiling point of methyl alcohol, however, mercuration is complete and compound (III) is produced. The nitro compounds undergo simple addition in the cold to give a carboxylic body corresponding to formula (II) and on heating they go over into the anhydride corresponding to formula (I). No mercuration of the nucleus has been found to be possible.

The simple addition products readily dissolve in alkali and when the solutions are acidified with dilute sulphuric acid the anhydride forms are precipitated. When dilute hydrochloric acid is employed instead, the addenda are removed and the original acids are regenerated if they are *trans* bodies and if they are *cis* they undergo inversion during this process and yield the corresponding *trans* compounds. In those cases where acetoxymercuri groups have entered the nucleus the mineral acids yield sulphatomercuri- and chloromercuri-compounds. By the action of alkali and hydrogen sulphide the entire mercury can be removed and β-methoxydihydro compounds (IV) are obtained as usual. These are found to be stable in the presence of dilute acid and alkali solutions.

At this stage it will be useful to sum up the results in regard to the action of mercuric acetate on cinnamic and substituted cinnamic acids in methyl alcoholic solutions. Some of them form mercuric salts which separate in a crystalline form immediately. In general these subsequently undergo change into addition and mercuration products. All the acids undergo addition at the double bond without any exception. Mercuration of the benzene ring with mercuric acetate does not seem to be a powerful reaction and hence the possibility of mercuration is dependent upon the nature of the substituent groups. Cinnamic acid itself does not undergo mercuration whereas the presence of a methoxyl or hydroxyl substituent renders mercuration possible owing to their activating influence on the benzene ring. On the other hand, the nitro group is known to have a powerful deactivating influence and consequently, the nitrocinnamic acids are not capable of mercuration. In the case of the methyl ethers of nitrocoumarinic and coumaric acids the nitro group counteracts effectively the activating influence of the methoxyl so that the compounds do not undergo mercuration. The hydroxyl

group however is evidently more powerfully activating and the hydroxy-cinnamic acids even in the presence of a nitro-group get mercurated.[3]

The hydroxy- and methoxy-cinnamic acids do not yield the anhydro form on treatment in the cold or on heating with mercuric acetate in methyl alcoholic solution. The same behaviour is exhibited by nitrocoumaric acid also. If, however, a nitro group is present along with a methoxyl the carboxyl form is produced in the cold and the anhydride on heating. On the other hand, cinnamic and nitrocinnamic acids yield the anhydride forms only. Hence the elimination of a molecule of acetic acid so as to form the anhydrides from the initial addition products seems to be definitely influenced by the nature of the substituents. Groups like the hydroxyl and methoxyl prevent this elimination whereas the nitro group facilitates the process. They probably play their part through their capacity to regulate the ionisation of the hydrogen of the carboxyl group present in these acids.



(V)

The β-methoxydihydrocinnamic acids that are produced by the action of hydrogen sulphide on the solutions of the mercuric compounds in alkali are quite stable in the presence of dilute alkali and acid when no hydroxyl group is present in the nucleus. Ordinary coumaric acid yields β-methoxy-melilotic acid which is rather unstable whereas its nitro and methyl derivatives do not produce the corresponding methoxy-dihydro-compounds.[3]

Experimental.

The methods for the preparation of the methyl ethers of coumarinic and coumaric acids were given in detail by Rangaswami and Seshadri.[4] The use of sodium ethoxide and methyl iodide has not proved quite consistent in yielding pure samples of the methyl ether of coumarinic acid. Frequently the isomeric ether of coumaric acid has contaminated it. The following method using aqueous alcoholic sodium hydroxide and dimethyl sulphate has been found to be the best. Coumarin (4 g.) was dissolved in a boiling mixture of methanol (50 c.c.) and 20% aqueous sodium hydroxide (50 c.c.) After cooling to about 50° C., it was treated with dimethyl sulphate (30 c.c.) which was added in small amounts with vigorous shaking. The mixture was then shaken for half-an-hour and 20% aqueous sodium hydroxide (100 c.c.) and more dimethyl sulphate (15 c.c.) again added. The reaction

was completed by heating the mixture for one hour. On acidifying the cooled solution the required methyl ether separated as a mass of crystals. It was purified by dissolving in sodium carbonate and reprecipitating with acid and finally recrystallising from alcohol. It melted at 91–2°, and the yield was 3·8 g.

The corresponding *trans* ether is best obtained by starting with coumaric acid which is easily produced from coumarin by the action of mercuric oxide in the presence of alkali. Coumaric acid is then methylated by the above procedure.

The ethers derived from nitrocoumarin were obtained by methods already described.[4]

Action of Mercuric Acetate.

1. On the Methyl Ether of Coumarinic Acid :

α-Acetoxymercuri-β : 2-dimethoxydihydrocinnamic acid (II).—When equimolecular proportions of the methyl ether of coumarinic acid (1·8 g. in 10 c.c. of methyl alcohol) and mercuric acetate (3·2 g. in 20 c.c. of methyl alcohol) were mixed in methyl alcoholic solution there was no immediate precipitation of the mercuric salt. But after some minutes a colourless crystalline solid began to separate out and the reaction was complete in about 4 hours. The additive compound was filtered and washed thoroughly with methyl alcohol. It was found to decompose at 204° and was completely soluble in dilute sodium hydroxide. (Found : Hg, 42·7 ; $C_{13}H_{16}O_6Hg$ required Hg, 42·8%.) The anhydride (I) was obtained by dissolving the above compound in dilute sodium hydroxide and acidifying the solution with aqueous sulphuric acid. It decomposed at 189° and was found to be quite different from the above acid form. (Found : Hg, 48·3 ; $C_{11}H_{12}O_4Hg$ required Hg, 49·1%.) If, on the other hand, the alkaline solution was treated with aqueous hydrochloric acid the addenda were removed accompanied by geometrical inversion and the resulting product was identical with the methyl ether of coumaric acid melting at 183°.

α : 3 : 5-Triacetoxymercuri-β : 2-dimethoxydihydrocinnamic acid.—When more than three molecular proportions of mercuric acetate (12·5 g.) were employed and heated with the methyl ether of coumarinic acid (1·8 g.) in methyl alcoholic solution for about 20 hours, addition and mercuration were effected and a good yield of the product (III) was obtained. It was advisable to discard the first precipitate formed during the first half hour. After repeatedly washing with methyl alcohol, the substance appeared as a white powder and decomposed at 220–21°. (Found : Hg, 60·6 ; C, 20·1 ; $C_{17}H_{20}O_{10}Hg_3$ required Hg, 61·0 ; C, 20·7%.) When a solution of the substance in

aqueous sodium hydroxide was acidified with hydrochloric acid a colourless precipitate of 3 : 5-dichloro-mercuri-2-methoxy-cinnamic acid was obtained. It decomposed at 216°. (Found : Hg, 62·5 ; $C_{10}H_8O_3Hg_2Cl_2$ required Hg, 61·9%.) With dilute sulphuric acid, however, the addenda were not removed and a sulphatomercuri compound, decomposing at 226° was produced. (Found : Hg, 66·0 ; $C_{11}H_{10}O_4Hg_3SO_4$ required Hg, 66·6%.)

β : 2-*Dimethoxydihydrocinnamic acid.*—This was obtained from any of the above mercuri-compounds by the action of hydrogen sulphide on their solutions in aqueous alkali. After filtering off mercuric sulphide the solution was cooled in ice and acidified with dilute sulphuric acid. The acid was thus obtained as a crystalline solid which was easily soluble in the common organic solvents and was best crystallised from hot water. It appeared as hexagonal plates melting at 84–5° and was found to be identical with the compound obtained by Biilmann.[2] It was found to be stable in the presence of dilute alkali and acid.

2. *On the Methyl Ether of Coumaric Acid* :

No mercury salt was formed in this case also. With one molecular equivalent of mercuric acetate in cold methyl alcoholic solution the addition was comparatively slow and a precipitate began to appear after about 20 minutes. When the product was collected after 4 hours it was mostly the addition product which had undergone some mercuration also. No definite compound could be obtained. When more than three molecular proportions of mercuric acetate were employed and the mixture boiled for about 20 hours a triacetoxy-mercuri-compound identical with the one obtained from the *cis* ether was produced. It decomposed at 220°. (Found : Hg, 61·6 ; $C_{17}H_{20}O_{10}Hg_3$ required Hg, 61·0%.) The chloro- and sulphatomercuri-compounds were also identical. By the action of hydrogen sulphide the same *β* : 2-dimethoxydihydrocinnamic acid was produced.

3. *On the Methyl Ether of* 5-*Nitro Coumarinic Acid* :

β : 2-*Dimethoxy-α-acetoxymercuri*-5-*nitro-cinnamic acid* (*II*).—When a clear solution of this acid (2 g.) in methyl alcohol (40 c.c.) was mixed with a similar solution of mercuric acetate (3 g. in 20 c.c.) in the cold, there was an immediate precipitation of the mercuric salt of the acid as colourless needles decomposing at 141–42°. With aqueous sodium hydroxide it yielded mercuric oxide and from the filtered solution the original acid could be precipitated pure by acidification. The salt gradually underwent conversion into *β* : 2-dimethoxy-α-acetoxymercuri-5-nitro-cinnamic acid and the change was complete in 24 hours. The pale yellow solid had no definite crystal structure and it decomposed at 199°. (Found : Hg, 38·4 ; $C_{13}H_{15}O_8Hg$ N required Hg, 39·0%.)

The anhydride form (*I*).—After mixing the two solutions the mixture was heated on a water-bath for about 5 hours. The mercuric salt that was first precipitated rapidly dissolved and a deep yellow solid began to separate after about an hour. After the completion of the heating the solid was filtered and washed with methyl alcohol. It had no definite crystal structure and decomposed at 210°. It was different from the solid obtained in the cold, but could be obtained from it by dissolution in aqueous caustic soda and subsequent acidification with dilute sulphuric acid. (Found : Hg, 44·4 ; C, 29·0 ; $C_{11}H_{11}O_6HgN$ required Hg, 44·2 ; C, 29·1%.)

On acidifying an aqueous alkaline solution of the simple addition product or the anhydride with dilute hydrochloric acid geometrical inversion was produced and the methyl ether of 5-nitro-coumaric acid melting at 233° was obtained.

β : 2-*Dimethoxy*-5-*nitrodihydrocinnamic acid* was obtained from the alkaline solutions of the above addition compounds by the action of hydrogen sulphide. It crystallised from hot water in the form of spear heads and melted at 158°. (Found : N, 5·8 ; $C_{11}H_{13}O_6N$ required N, 5·5%.) It was easily soluble in all common solvents and was quite stable in the presence of alkali and acid.

4. On the Methyl Ether of 5-Nitro-Coumaric Acid :

The acid (2 g.) was dissolved in methyl alcohol (50 c.c.) and mixed with a solution of mercuric acetate (3 g. in 20 c.c.) to which a few drops of acetic acid had been added. A bulky precipitate separated out immediately and it gave all the reactions of the mercuric salt. (Found : Hg, 32·0 ; $C_{20}H_{16}O_{10}HgN_2$ required Hg, 31·1%.) It decomposed at 205°.

The mercuric salt underwent change slowly when it was not separated and after two days it was completely converted into the addition compound, α-acetoxymercuri-β : 2-dimethoxy-5-nitro-dihydrocinnamic acid. It was a pale yellow solid decomposing at 196° and identical with the one obtained from the methyl ether of nitro-coumarinic acid. On heating the mixture, however, on the water-bath for 5 hours the corresponding anhydride decomposing at 210° was easily produced. This was found to be identical with the compound obtained from the *cis* ether. Both the acid and anhydride gave rise to β : 2-dimethoxy-5-nitro-dihydrocinnamic acid when decomposed with hydrogen sulphide.

Summary.

The action of mercuric acetate on the methyl ethers of coumarinic and coumaric acids and their 5-nitro derivatives has been described. The first two undergo addition at the double bond as well as mercuration and the

compounds have the carboxylic formula. The nitrocoumarinic and coumaric acids undergo addition only and yield the carboxylic form in the cold and the anhydro form on heating. When the addenda are eliminated by the action of dilute hydrochloric acid the change is accompanied by geometrical inversion from *cis* to *trans*. The results so far obtained with cinnamic acid and its derivatives are reviewed.

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THE EXPANSION OF A FUNCTION IN A SERIES OF ASSOCIATED LEGENDRE FUNCTIONS.

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(Communicated by Prof. B. S. Madhava Rao.)

1. D. P. Banerji[1] has recently given a theorem for the expansion of an arbitrary function in a series of conal or toroidal functions. The object of this note is to give an extension of this theorem to the expansion of an arbitrary function in a series of $P_{-\frac{1}{2}+pi}^{m}(\cosh\psi) = K_p^m(\cosh\psi)$ or of $P_{n-\frac{1}{2}}^{m}(\cos\theta)$.

2. The function $K_p^m(\cosh\psi)$ is given by[2]

$$K_p^m(\cosh\psi) = P_{-\frac{1}{2}+pi}^{m}(\cosh\psi)$$

$$= \frac{2^{m+1}\sinh^m\psi}{\Pi(-\frac{1}{2})\,\Pi(-m-\frac{1}{2})}\int_0^{\psi}\frac{\cos pu\,du}{\{2(\cosh\psi-\cosh u)\}^{m+\frac{1}{2}}},$$

where $R(\frac{1}{2}-m)>0$ and the phase of $2\cosh\psi-2\cosh u$ is zero. (1)

Suppose now that the function $F(x)$ is expressible in the form

$$F(x) = \int_1^x \frac{u(\xi)\,d\xi}{(x-\xi)^{m+\frac{1}{2}}},$$

where $0 < m+\frac{1}{2} < 1$. (2)

This is Abel's integral equation and its solution is[3]

$$u(x) = \frac{\cos m\pi}{\pi}\cdot\frac{d}{dx}\int_1^x \frac{F(\xi)\,d\xi}{(x-\xi)^{\frac{1}{2}-m}}. \tag{3}$$

Assuming $F(x)$ to satisfy (2) and (3) we have

$$F(\cosh\psi) = \int_1^{\cosh\psi}\frac{u(\xi)\,d\xi}{\{\cosh\psi-\xi\}^{m+\frac{1}{2}}}$$

$$= \int_0^{\psi}\frac{u(\cosh\phi)\sinh\phi\,d\phi}{(\cosh\psi-\cosh\phi)^{m+\frac{1}{2}}}. \tag{4}$$

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[1] "The expansion of an arbitrary function in a series of conal or toroidal functions," The *Proc. of the Cambridge Phil. Soc.*, **34**, Part I, 30–32.

[2] Hobson, *The Theory of Spherical and Ellipsoidal Harmonics*, p. 270, formula (139).

[3] Whittaker and Watson, *Modern Analysis*, 4th ed., p. 229.

Expand $u\ (\cosh\phi)\ \sinh\phi$ as a Fourier series,

$$u\ (\cosh\phi)\ \sinh\phi = \sum_{p=0}^{\infty} a_p \cos p\phi \qquad (0 \leqslant \phi \leqslant \pi)$$

and we have the expansion

$$\mathrm{F}\ (\cosh\psi) = \sum_{p=0}^{\infty} a_p \cdot \int_0^{\psi} \frac{\cos p\phi\ d\phi}{\{\cosh\psi - \cosh\phi\}^{m+\frac{1}{2}}}$$

$$= \sum_{p=0}^{\infty} a_p \frac{\Pi\left(-\tfrac{1}{2}\right)\Pi\left(-m-\tfrac{1}{2}\right)}{\sqrt{2}\ (\sinh\psi)^m}\ \mathrm{K}_p{}^m\ (\cosh\psi)$$

$$= \sum_{p=0}^{\infty} \mathrm{A}_p\ \mathrm{K}_p{}^m\ (\cosh\psi), \qquad (5)$$

where $-\frac{1}{2} < m < \frac{1}{2}$,

$$\mathrm{A}_p = \frac{\sqrt{2}}{\pi} \frac{\Pi\left(-\tfrac{1}{2}\right)\Pi\left(-m-\tfrac{1}{2}\right)}{(\sinh\psi)^m} \int_0^{\pi} u\ (\cosh\ \phi)\ \sinh\phi \cdot \cos p\phi\ d\phi$$

$$(p \geqslant 1),$$

and $$\mathrm{A}_0 = \frac{1}{\sqrt{2}\ \pi} \frac{\Pi\left(-\tfrac{1}{2}\right)\Pi\left(-m-\tfrac{1}{2}\right)}{(\sinh\psi)^m} \int_0^{\pi} u\ (\cosh\phi)\ \sinh\phi\ d\phi.$$

3. Again, we have[4]

$$\mathrm{P}_{n-\frac{1}{2}}^{-m}\ (\cos\theta) = \frac{2^{-m+1}}{\Pi\left(-\tfrac{1}{2}\right)\Pi\left(m-\tfrac{1}{2}\right)} (\sin\theta)^{-m} \int_0^{\theta} \frac{\cos n\phi\ d\phi}{\{2\ (\cos\phi - \cos\theta)\}^{m+\frac{1}{2}}},$$

where $\mathrm{R}\left(\frac{1}{2} - m\right) > 0$. (6)

Proceeding as in the previous article, we get

$$\mathrm{F}\ (\cos\theta) = \sum_{n=0}^{\infty} b_n\ \mathrm{P}_{n-\frac{1}{2}}^{-m}\ (\cos\theta) \qquad (0 \leqslant \theta \leqslant \pi) \qquad (7)$$

where $|\ m\ | < \frac{1}{2}$,

$$b_n = \frac{\sqrt{2}}{\pi} \frac{\Pi\left(-\tfrac{1}{2}\right)\Pi\left(m-\tfrac{1}{2}\right)}{(\sin\theta)^{-m}} \cdot \int_0^{\pi} u\ (\cos\phi) \sin\phi \cos n\phi\ d\phi \qquad (n \geqslant 1)$$

and $$b_0 = \frac{1}{\sqrt{2}\pi} \frac{\Pi\left(-\tfrac{1}{2}\right)\ \Pi\ \left(m-\tfrac{1}{2}\right)}{(\sin\theta)^{-m}} \int_0^{\pi} u\ (\cos\phi) \sin\phi\ d\phi.$$

Banerji's theorems[5] can be obtained by putting $m = 0$ in (5) and (7) above.

[4] Hobson, *loc. cit.*, p. 267, formula (128).

[5] *Loc. cit.*, formulæ (5) and (7).

THE FLAME SPECTRA OF SOME ALIPHATIC HALIDES—Part II.

(2) Ethyl Bromide.

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Received April 15, 1938.
(Communicated by Prof. B. S. Madhava Rao.)

1. *Introductory.*

In a previous communication,[1] the author had pointed out that the spectrum of the flame of methyl iodide included, besides the well-known bands of C_2, CH and HO, two systems of bands which had not been noticed before. From various considerations they were attributed to IO. The present paper is an account of the spectrum of the flame of ethyl bromide. Though methyl bromide is the substituted halide compound of methane corresponding to methyl iodide, ethyl bromide was selected because its flame was more convenient to handle in giving long exposures. Moreover, the introduction of an extra group of CH_2 in CH_3Br would not be expected to affect the spectrum of CH_3Br very materially.

The new flame bands of methyl iodide were degraded towards the red. Similar bands have been obtained in the flame of ethyl bromide; though there are indications of several bands, measurements could be taken on only a few because a strong continuous spectrum overruns the region of observation, which makes the location of band heads difficult.

2. *Experimental Details.*

Ethyl bromide burns satisfactorily when oxygen supports its combustion. A gentle current of air was bubbled through a quantity of ethyl bromide (Kahlbaum) placed in a small wash bottle and this mixture flowed through the inner nozzle of a two-way blow-pipe, while through the outer nozzle was blown a current of oxygen from a compressed gas cylinder. It was found that ethyl bromide burnt much more steadily if the usual arrangements of burning a gas were reversed. The height of the flame was about 4 cm. of which the inner cone occupied about 1·5 cm.; the breadth was about 0·5 cm. The outer cone presented an orange-yellow appearance,

while the inner cone was bluish green. As in the flame of methyl iodide, the middle portion of the outer cone just above the inner cone was focussed on the slit of the spectrograph. Since the flame was small in height as compared with the methyl iodide flame, the upper portion of the inner cone shot up every now and then even when the middle portion was carefully focussed on the slit. This resulted in the appearance of the C_2 and CH bands which being in the same region as the flame bands somewhat interfered with the measurements. A Smithells' flame separator would have obviated this trouble but since the flame could burn steadily only when oxygen surrounded the ethyl bormide vapour, the flame separator could not be used.

Exposures of nearly 40 hours were found sufficient to photograph most of the fairly intense bands on the Hilger's E_1 spectrogrpah. A concave mirror behind the flame, as in the case of methyl iodide, materially reduced the time of exposure. There were indications of several faint bands between λ 3300 and λ 3800, but when exposure was increased, the continuous spectrum masked them completely. The same difficulty was experienced when exposures were given on the Hilger's E_2 spectrogrpah. Hence, unfortunately no measurements could be taken on these band heads. Iron arc was the source of reference lines and the spectra were taken on the Ilford's Special Rapid Panchromatic plates.

3. *Description of the Spectrum.*

The spectrum of the outer cone of the flame of ethyl bromide consists of the well-known water vapour bands at λ 3064, λ 2811, etc. ; the new flame bands corresponding to the methyl iodide flame bands and a group of diffuse bands in the yellow-orange which have been identified with the Br_2 bands as shown below. Since the inner cone always shoots up, the C_2 and the CH bands also appear which obscure some of the flame bands. A strong continuous spectrum overruns the whole region of observation which makes the exact location of band heads somewhat difficult. With the dispersion employed there were no distinct signs of the resolution of the structure of the individual bands (Plate VIII *a* and *b*).

4. *Bands in Yellow-Orange.*

The bands in the yellow-orange are indistinct and very close together superimposed by a strong continuous spectrum. They are degraded towards the red. By measurements they have been identified with the Br_2 bands. Kitagawa[2] had observed that a group of bands appeared in the same region when hydrogen mixed with bromine was burnt. From measurements he had concluded that these bands were identical with the Br_2 bands investigated

by Brown. If the author's measures are compared with those of Kitagawa, they agree closely as will be seen from Table I. Hence the bands in the yellow-orange have to be attributed to Br_2. Their appearance can be gathered from Plate VIII *c*.

TABLE I.

ν	Kitagawa λ	Vaidya λ
17852 (0)	5600	Very faint
17762 (1)	5628·4	
17663 (1)	5659·9	
17563 (1)	5692·2	
17467 (2)	5727·3	5723·9 (3)
17377 (1)	5753·1	5752·5 (2)
17354 (2)	5760·7	
17254 (3)	5794·1	5788·1 (3)
17230 (1)	5802·2	
17146 (4)	5830·6	5822·5 (3)
17039 (5)	5867·1	5862·4 (4)
16916 (5)	5909·9	5910·0 (2)
16827 (3)	5941·2	5935·8 (2)
16781 (3)	5957·4	5956·9 (2)
16712 (4)	5982·0	
16596 (4)	6023·8	6019·9 (2)
16508 (2)	6055·9	
16466 (4)	6071·4	6074·8 (4)
16398 (5)	6096·6	6093·5 (2)
16334 (4)	6120·5	6128·7 (3)

TABLE I—(*Contd.*)

ν	Kitagawa λ	Vaidya λ
16276 (6)	6142·3	
16209 (5)	6167·7	6162·3 (2)
16155 (7)	6188·3	6181·8 (4)
16072 (3)	6220·2	6210·8 (3)
16021 (8)	6240·1	6240·1 (1)
15961 (6)	6263·5	6263·1 (1)
15893 (9)	6290·3	6291·7 (3)
15836 (8)	6312·9	6311·9 (2)
15763 (7)	6342·2	6342·1 (2)
15719 (9)	6359·9	6365·8 (3)
15605 (6)	6406·4	6406·9 (3)
15572 (10)	6417·1	6413·7 (4)
15443 (10)	6473·6	6460·4 (3)
15305 (8)	6532·0	
15269 (8)	6547·4	6546·2 (3)
15163 (7)	6593·1	6595·7 (2)
15135 (7)	6605·3	
14995 (5)	6667·0	6662·9 (1)
14978 (5)	6674·6	
14857 (3)	6726·9	
14823 (3)	6744·4	Beyond the range of the author's measurements
14700 (1)	6800·4	
14541 (0)	6875·2	

5. Wave-lengths and Analysis of the Ethyl Bromide Bands.

Though there are indications of several bands only a few could be satisfactorily measured because most of them are faint and have ill-defined heads. If the exposure is increased or the spectrum is taken on an instrument of smaller dispersion, the continuous spectrum is strengthened which masks them. Hence, unfortunately no measurements could be taken on bands between λ 3300 and λ 3800. The first member of each sequence has a defined head while the rest appear as broad patches of uniform intensity. The analysis is therefore somewhat incomplete. Attempts are being continued to get brighter sources of the bands. A discharge in vacuum-tubes through various bromine compounds as well as pure bromine along with oxygen did not yield these bands. The results of the measurements are given in Table II.

TABLE II.

λ	ν
4029·2 (3)	24813
4065·8 (4)	24589
4147·0 (5)	24108
4178·1 (5)	23929
4269·8 (7)	23415
4399·4 (7)	22725
4430·2 (4)	22567
4534·2 (8)	22049

The bands have been arranged in a v', v'' scheme in the usual way as shown in Table III. Since the analysis is incomplete, no attempt has been made to asign the exact values of quantum numbers. From the appearance of bands it does not seem probable that this system extends very far on the short wave-length side; the unmeasured bands between λ 3300 and λ 3800 probably belong to a different system but interrelated as the systems A and B of the methyl iodide flame bands. Hence w_e'' should not be very different from 710 cm.$^{-1}$ Since most of the molecules in the excited state would be in the energy level $v' = 0$, w_e' would be of the order of 520 cm.$^{-1}$ Thus, this band system would correspond to the system B of the methyl

iodide flame bands where w_e'' was 671 cm.$^{-1}$ while that of A 536 cm.$^{-1}$ The bands corresponding to the system A would probably be those between λ 3300–λ 3800.

TABLE III.

	v''	$v''+1$	$v''+2$	$v''+3$	$v''+4$
v'	24813(3) *705* 24108 (5)	*693*	23415 (7) *690*	22725 (7) *676*	22049 (8)
			514		*518*
$v'+1$			23929 (5)	*	22567 (4)
$v'+2$				*	†

* Obscured by CH band

† Obscured by C_2 band

Unclassified

λ	ν
4065·8 (4)	24589

6. The Emitter of the Bands.

The experimental evidence is strongly in favour of Br–O being the emitter, for the ethyl bromide flame bands occur under circumstances under which the IO bands were observed in the methyl iodide flame. That the bands under discussion are not due to any intermediate compound formed by the interaction of bromine liberated from ethyl bromide and the metallic burner was decided by examining the spectrum of the flame when ethyl bromide was burning at a pyrex jet. Under these circumstances the identical bands were obtained.

The separations of successive band heads in the analysis do not seem to be related to the known band systems of Br_2. The analysis also shows that w_e'' is about 710 cm.$^{-1}$ which is greater than $w_e'' = 671$ cm.$^{-1}$ of IO. This is to be expected if the bands under discussion are attributed to BrO.

The author is very grateful to Sir C. V. Raman for giving the necessary facilities. He is also thankful to Dr. P. C. Guha, Head of the Organic Chemistry Department, Indian Institute of Science, for sparing Kahlbaum's ethyl bromide used in the present investigation.

Summary.

Bands corresponding to IO, occurring in the flame of methyl iodide, have been obtained in the flame of ethyl bromide. The experimental evidence and the vibrational analysis are in favour of BrO being the emitter of these bands. Also a large number of bands in the yellow-orange have been shown to be due to Br_2.

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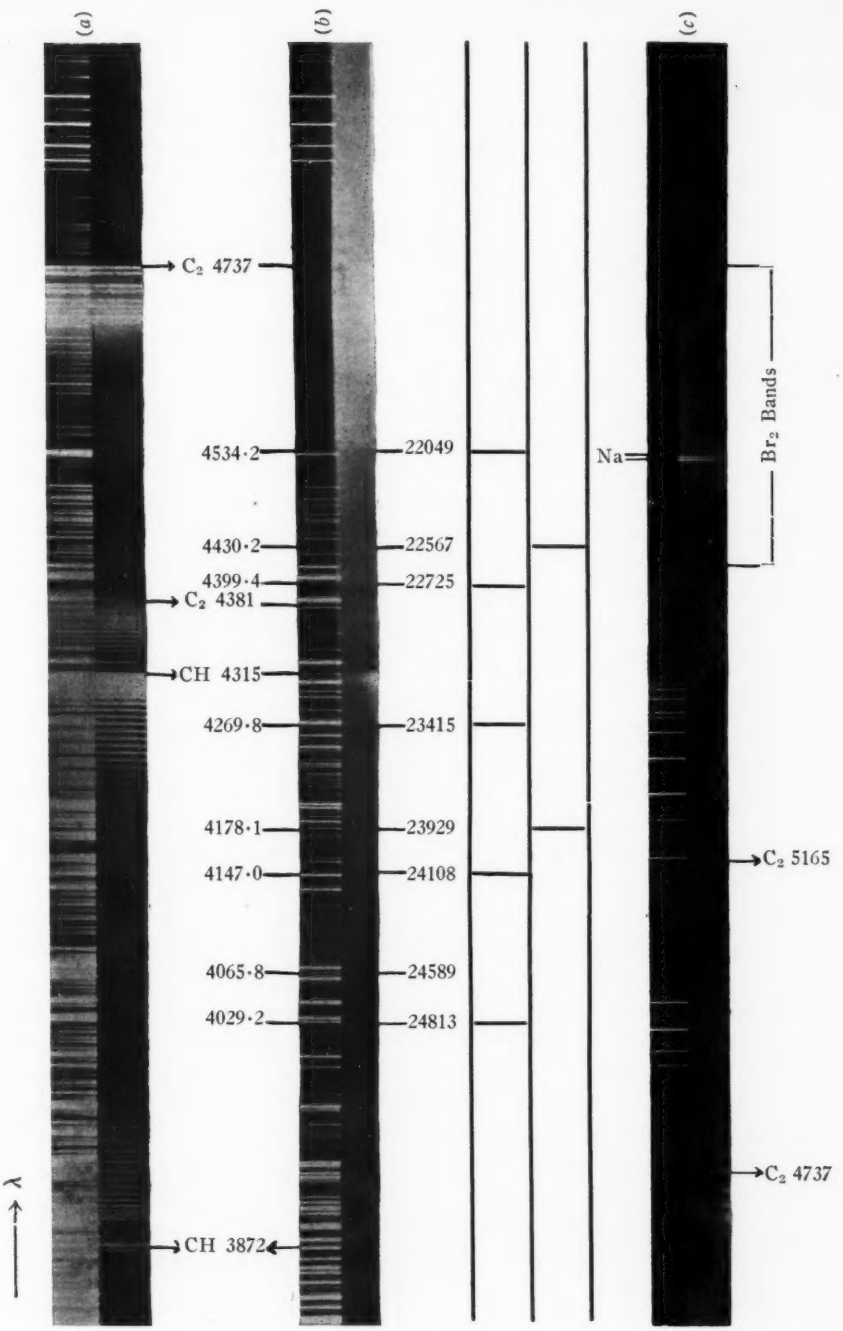
(a)
(b)
(c)
C_2 4737
4534·2
22049
4430·2
22567
4399·4
22725
C_2 4381
CH 4315
4269·8
23415
4178·1
23929
4147·0
24108
4065·8
24589
4029·2
24813
CH 3872
λ
Na
Br_2 Bands
C_2 5165
C_2 4737

VISCOSITY AND JELLY FORMATION OF CONCENTRATED SOLS OF HYDROUS STANNIC OXIDE.

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(Communicated by Mr. S. Bhagavantam.)

1. *Introduction.*

In a previous publication Dhar and Varadanam (1936) have described the "Preparation and properties of highly concentrated stannic hydroxide sols". It was pointed out in that connection that the behaviour of hydrous stannic oxide was different from that of the concentrated hydrous oxide sols of Fe, Al, Cr, etc. The viscosities of the colloidal stannic oxide are relatively small and the sols do not set to jellies even when dialysed for 10 to 15 days. With a view to throw further light on the question, the author has studied the viscosities of sols of hydrous stannic oxide in greater detail.

2. *Variation of the Electrolyte Content and Viscosity with Dialysis.*

The method of the preparation of the sol is the same as that adopted by Dhar and Varadanam (*loc. cit.*). Part of the sol is subjected to "cold dialysis" (temperature about 27° C.) using a parchment bag that has previously been treated with dilute nitric acid and kept immersed for two days in distilled water. Another portion of the same sol is subjected to "hot dialysis" in a well-regulated thermostat kept at 39°·8 C. From time to time samples of the sol from the two bags are transferred to well-cleaned jena glass bottles for investigation. The dialysate is replaced by fresh water once a day in the case of dialysis at the laboratory temperature and once in 12 hours when the process is carried out at the higher temperature.

The electrolytes present in the sol are ammonium hydroxide and chloride ion. The chloride is estimated gravimetrically as silver chloride. It is found that it dialyses out within a short time. The main impurity after a certain stage is therefore only ammonia.

To know the amount of the electrolyte in the sol after "cold" or "hot dialysis", a known amount of the sol is directly distilled with excess of sodium hydroxide solution and the "total" ammonia evolved is absorbed in acid and is estimated using the Duboscq colorimeter with 0·0001 N. ammonium chloride as the standard for comparison. The "free" ammonia in

a sol is estimated by first coagulating the sol with barium chloride and diluting the liquid to a known volume and keeping it for 24 hours for the attainment of equilibrium. A known amount of the supernatant liquid is then pipetted out and the ammonia estimated as before. The difference between the "total" ammonia and the "free" ammonia for the same amount of the sol will give a measure of the ammonia adsorbed by the micelles. It cannot be said that this method would give absolute values of the adsorbed ammonia. Since, however, the same method was followed throughout this investigation, the results obtained will give an indication of the relative amounts of adsorbed ammonia for the same sol at different stages of dialysis.*

The results are given in Tables I and II.

TABLE I. *Cold Dialysis.*

(A) Period of dialysis in days	(B) No. of g. of stannic oxide per 100 g. of the sol	(C) "Total" ammonia in mg. per 100 g. of the sol	(D) "Free" ammonia in mg. per 100 g. of sol	Difference between (C) & (D) in mg. per 100 g. of the sol	Ammonia in mg. associated with micelles in 1 g. of stannic oxide	Chloride as impurity per 100 g. of the sol	Viscosity of the sol at 39°·8C. (viscosity of water taken at that temp. as 0·0066.)
3	6·28	158·68	88·38	70·30	11·90	30·70	0·0090
5	5·92	135·50	75·43	60·07	10·15	Traces	0·0100
7	5·65	107·86	55·09	52·77	9·34	..	0·0102
10	5·37	96·60	48·00	48·60	9·05	..	0·0104
15	4·87	69·06	32·52	31·54	6·49	..	0·0109
18	4·66	60·35	32·80	27·55	5·92	..	0·0111
25	4·26	49·51	27·99	21·52	5·06	..	0·0116
39	3·74	38·08	21·15	16·93	4·59	..	0·0119
75							Viscous
80	5·88	39·25	13·50	25·75	4·38		Stiff Jelly

* There appears to be no reliable method for determining the amount of adsorbed electrolytes. The ultra-filtration method used by Verwey and Kruyt (1933) was criticised by Weiser and Milligan (1936).

TABLE II. *Hot Dialysis.*

(A) Period of dialysis in hours	(B) No. of g. of stannic oxide per 100 g. of the sol	(C) "Total" ammonia in mg. per 100 g. of the sol	(D) "Free" ammonia in mg. per 100 g. of sol	Difference between (C) & (D) in mg. per 100 g. of the sol	Ammonia in mg. associated with micelles in 1 g. of stannic oxide	Chloride as impurity per 100 g. of the sol	Viscosity of the sol at 39°.8C. (viscosity of water taken at that temp. as 0·0066).
6	6·61	208·00	135·10	72·90	11·05	39·93	0·0082
14	7·86	185·60	114·40	71·20	9·00	Traces	0·0169
21	7·44	163·66	100·76	62·90	8·45	..	0·0363
27	7·88	131·10	81·94	49·16	6·24	..	Viscous
30	7·84	122·20	75·83	46·37	5·93	..	Highly Viscous
41	11·27	141·80	75·46	66·34	5·89	..	Stiff Jelly

3. Study of the Variation of Particle Size and Shape with Dialysis.

This is carried out by employing the optical method recently developed by R. S. Krishnan (1934). The method consists in measuring the depolarisation factors of scattered light using alternately unpolarised light, horizontally polarised light and vertically polarised light. The terms ρ_u, ρ_h, ρ_v occurring in the paper signify respectively the depolarisation factors of scattered light when the incident beam is unpolarised, when it is horizontally polarised and vertically polarised. The depolarisation is expressed as the ratio of the intensity of the weak component to that of the strong component irrespective of whether the horizotal component is weaker or stronger than the vertical component. The experimental arrangements are the same as those described by Jogarao (1936). The sol used for this part of the work was prepared fresh by the same method and under the same conditions, as employed for the sol used for the ammonia estimations.

The readings are taken when the sol is freshly prepared and also during different stages of dialysis (cold as well as hot). The results are given in Tables III and IV. To eliminate the error due to concentration fluctuation, the concentration of the sol in one case (Table III) is adjusted to 3 grams of stannic oxide per 100 grams of the sol, by diluting with water and the viscosities are again found out. The values are given in the last column.

TABLE III. *Cold Dialysis.*

Period of dialysis in days	No. of g. of stannic oxide per 100 g. of the sol	Size and shape of the particles			Viscosity at 39°·8 C. of the sol for the same concentration	Viscosity at 39°·8 C. for 3% concentration of stannic oxide of the sol
		ρ_u	ρ_h	ρ_v		
0	9·02	0·01	1·0	0·005	0·0115	0·0077
2	8·20	0·01	1·0	0·005	0·01328	
5	7·58	0·031	1·0	0·011	0·0147	0·0090
12	6·92	0·031	0·97	0·011	0·0298	0·0133
21	5·77	0·031	0·78	0·011	0·0248	0·0132
30	5·00	0·031	0·70	0·011	0·0196	0·0126
41	4·43	0·031	0·59	0·011	0·0178	0·0136
51	4·15	0·031	0·49	0·011	0·0175	0·0139
61	3·90	0·031	0·49	0·011	0·0190	0·0158
73	3·75	0·031	0·49	0·011	Highly viscous	
85	6·25	0·031	0·49	0·011	Set to a jelly	

TABLE IV. *Hot Dialysis.*

Period of dialyss in hours	No. of g. of stannic oxide per 100 g. of the sol	Size and shape of the particles			Viscosity at 39°·8 C.
		ρ_u	ρ_h	ρ_v	
0	7·74 (Original diluted)	0·111	1·0	0·005	0·0099
4	7·55	0·031	0·84	0·015	0·0099
8	7·70	0·031	0·78	0·015	0·0256
13	7·87	0·031	0·76	0·015	0·1193
17	7·98	0·031	0·70	0·015	0·5117
23	8·50	0·031	0·55	0·011	Highly Viscous
30	9·72	0·028	0·49	0·011	Set to a jelly

4. Discussion of Results.

In the case of silicic acid sol several workers have concluded that the ageing of the sol by dialysis is accompanied by the formation of highly polymerised colloidal particles. The results obtained in the present investigation on hydrous stannic oxide sol lead to the same conclusion. The value of ρ_h becomes less than unity with prolonged dialysis showing that the particle size is rapidly increasing. The author's work also indicates that the process of ageing is slower in " cold dialysis " than in " hot dialysis ". From Tables III an IV it is seen that in " cold dialysis " the value of ρ_h falls from 1 to 0·49 in 51 days whereas in " hot dialysis " a similar change is obtained in only 30 hours.

It will be of interest to note that during this process of ageing by dialysis the amount of " adsorbed " or " bound " ammonia per gram of stannic oxide also decreases with the increase in the time of dialysis until the sol sets to a jelly. Further it has been observed that such purification of the sol is more rapid in "hot dialysis" (39°·8 C.) than when the dialysis is carried out at the room temperature (27° C.). The method of " hot dialysis " is commonly adopted for hastening the purification of hydrous oxide sols. Marks Neidle (1917) observes in experiments on hydrous oxide sols of chromium, iron and aluminum, that one can obtain by " hot dialysis " in ten hours, sols of purity which would require some months of " cold dialysis ". It may be noted that a small difference of temperature of 10 to 20° C. is known to considerably influence the rate of purification by dialysis. Thus Neidle observed in the case of hydrous chromic oxide that the same degree of purity as is obtained in 34 hours at 75° to 80° C. can be secured in ten hours when the dialysis is carried out near the boiling point. A sol of this purity is got in 73 days of " cold dialysis ".

Viscosity determinations have also been made with progressive dialysis. In " cold dialysis " (Table III) though there is a considerable increase in the size of the particles as the values of ρ_h indicates, yet there is no corresponding increase in viscosity. In "hot dialysis" on the other hand there is a considerable increase in viscosity as well as an increase in particle size with prolonged dialysis. It seems therefore that an increase in particle size does not necessarily result in an increase of viscosity as supposed by Desai and collaborators (1936). According to the Einstein formula, mere aggregation of molecules or colloidal particles to form larger compact massive particles or micelles has no effect on viscosity. Bancelain (1911) working with gamboge and mastic sols and Oden (1912) with sulphur sols have found that the relative viscosities are indepenent of particle size, assuming that

the micelles are spherical. James W. McBain and M. E. Laing McBain (1937) have arrived at a similar conclusion.

5. Conclusions and Summary.

1. Krishnan's optical method has been employed to follow the changes in particle size in hydrous stannic oxide sol with the progress of dialysis.

(*a*) Since $\rho_h = 1$ for the undialysed sol and continues to be at this value for the first five days of dialysis in the cold, we may conclude that the particles are almost of molecular dimensions in the earlier stages of dialysis.

(*b*) The value of ρ_h becomes less than one (from 1 to 0·49) with increased dialysis showing that the particle size has increased.

(*c*) This increase in particle size is more rapid in "hot dialysis" than in "cold dialysis".

(*d*) ρ_v in both "hot" and "cold dialysis" is very low at all stages and is not much more than the error that may be attributed to a convergence of the incident beam. This indicates that the sol is made up of spherical particles.

2. The adsorbed ammonia per gram of stannic oxide has been found to decrease rapidly at first and slowly thereafter with progress of dialysis.

3. Viscosity determinations have also been made at different stages of dialysis and it is concluded that the increase of particle size has no correlation with the variation of viscosity.

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BIQUATERNIONS IN BORN'S ELECTRODYNAMICS.

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1. Introduction.

The non-linear Electrodynamics initiated by Born is, on account of its rich mathematical formalism, capable of several types of representation. Restricting ourselves to the case of complex representations we might mention those of Schrödinger and P. Weiss. In a previous paper[1] I have considered in detail the former representation and investigated its relationship with the ordinary representation in the earliest form of Born's field theory. I have also shown elsewhere[2] that the complex formalism of Weiss arises quite naturally when we introduce generalised action-functions in Born's Electrodynamics without the restriction that the invariant $G = 0$. This paper is concerned with a third type of complex representation by biquaternions given by Watson[3] for the case of Born's original field theory. He constructs four biquaternions corresponding to the four pairings $(\vec{B}, \vec{E})$, $(\vec{D}, \vec{H})$, $(\vec{D}, \vec{B})$ and $(\vec{E}, \vec{H})$ of the field vectors, obtains their tensors and the relations between primary and secondary field vectors in the first case. I shall indicate briefly how some of his results follow at once from the results in I. I also give in this paper more symmetrical representations by biquaternions such that

(i) each biquaternion is expressed purely in terms of the primary field vectors in all four cases.

(ii) The tensors of the four biquaternions should be the four functions $I, +1$, $H + 1$, $U + 1$, $V + 1$ corresponding to the above four pairings.

(iii) the relations between primary and secondary field vectors may be exhibited in all the four cases.

Next, I show that it is also possible to set up a biquaternion representation corresponding to Born's action-function in the generalised Infeld form

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[1] *Proc. Ind. Acad. Sci.*, (A), 1936, **4**, 575—referred to as I.

[2] *Ibid.*, —referred to as II.

[3] Watson, *Trans. Roy. Soc. Canada* (Sec. 3), 1936, **30**, 105—referred to as III.

of a sum of Lagrangian and Hamiltonian. Finally, I indicate a biquaternionic interpretation of the fundamental parameters introduced in the generalised action-function theory (of II). The notation will be the same as in I, II and III unless it differ from I.

2. Watson's Biquaternions.

The four biquaternions given by Watson are

$$q = 1 + i\,\mathrm{G} + \vec{\mathrm{B}} - i\,\vec{\mathrm{E}} \tag{2, 1}$$

$$q_1 = (1 + \mathrm{G}^2)/\mathrm{T}q + \vec{\mathrm{H}} - i\,\vec{\mathrm{D}} \tag{2, 2}$$

$$q_2 = (1 + \vec{\mathrm{D}})\,(1 + \vec{\mathrm{B}})/(1 + \mathrm{G}^2) \tag{2, 3}$$

$$q_3 = (1 + i\,\vec{\mathrm{H}})\,(1 - i\,\vec{\mathrm{E}}) \tag{2, 4}$$

with their tensors

$$\mathrm{T}q = \mathrm{L} + 1 \tag{2, 1 a}$$

$$\mathrm{T}q_1 = 1 - i\,\mathrm{Q} \tag{2, 2 a}$$

$$\mathrm{T}q_2 = \mathrm{U} + 1 \tag{2, 3 a}$$

$$\mathrm{T}q_3 = \sqrt{1 + \vec{\mathrm{B}}^2 + \vec{\mathrm{D}}^2 + \mathrm{G}^2} \tag{2, 4 a}$$

[It is doubtful if the value of $\mathrm{T}q_3$ is correct. It ought to be $(\mathrm{V} + 1)\ \sqrt{1 + \mathrm{G}^2}$.] Writing $q = r + i\,t = \gamma + i\delta$, where

$$\left.\begin{aligned} r &= 1 + \vec{\mathrm{B}}\,;\, t = \mathrm{G} - \vec{\mathrm{E}} \\ \gamma &= 1 - i\vec{\mathrm{E}}\,;\, \delta = \mathrm{G} - i\,\vec{\mathrm{B}} \end{aligned}\right\}$$

the relations connecting $\vec{\mathrm{D}}$, $\vec{\mathrm{H}}$ with $\vec{\mathrm{B}}$, $\vec{\mathrm{E}}$ are obtained as

$$\left.\begin{aligned} \vec{\mathrm{D}} &= \tfrac{1}{2}\{\alpha, r\} \\ i\vec{\mathrm{H}} &= \tfrac{1}{2}\{\beta, \gamma\} \end{aligned}\right\} \tag{2, 5}$$

where $\alpha = \dfrac{\mathrm{K}t}{\mathrm{T}q}$; $\beta = \dfrac{\mathrm{K}\delta}{\mathrm{T}q}$ and $\{\ \}$ denotes the commutator, and $\mathrm{K}t$ the conjugate quaternion.

It might be pointed out how some of the results in Watson's paper are co-sequences of the equations in I. In fact, equations (9), (15), (16) of the former are identical with equations (32), (34), (40), respectively of the latter.

From equations (2, 1) – (2, 4) it is easily seen that q_1, q_2 and $\mathrm{T}q_3$ which are related to the $(\vec{\mathrm{D}}, \vec{\mathrm{H}})$, $(\vec{\mathrm{D}}, \vec{\mathrm{B}})$ and $(\vec{\mathrm{E}}, \vec{\mathrm{H}})$ representations involve G and $\mathrm{T}q$ which belong to the first representation $(\vec{\mathrm{B}}, \vec{\mathrm{E}})$. Also the tensors (2, 2 *a*) and (2, 4 *a*) do not represent the action-functions H + 1 and V + 1. Further

relations analogous to (2, 5) in the other three cases cannot be derived from the above biquaternion representations.

3. Alternative Biquaternions.

We shall first set up alternatives to (2, 1) – (2, 2). Let

$$p = q = 1 + iG + \vec{B} - i\vec{E} \tag{3, 1}$$

and,
$$p_1 = 1 - iQ + \vec{D} + i\vec{H}. \tag{3, 2}$$

The representation (3, 1) is the same as (2, 1) while (3, 2) corresponds to taking $(\vec{D}, \vec{H})$ as primary vectors. Operating on p, with the bi-scalar $(1 + iQ)/(H + 1)$ we get

$$q' = L + 1 + \vec{E} + i\vec{B} \tag{3, 3}$$

if we observe that

$$Tp_1 = H + 1 \tag{3, 2 a}$$

and,
$$(L + 1)(H + 1) = (1 + G^2) = (1 + Q^2). \qquad \text{[I, (18)]}$$

We can also immediately observe the relations between q, q_1 ; p_1, q'. In fact

$$\left.\begin{aligned} Tq' = Sq\,; \quad Tq' = Sq \\ Tq_1 = Sp_1\,; \; Tp_1 = Sq_1 \end{aligned}\right\} \tag{3, 4}$$

i.e., q_1, p_1 are so related that *the tensor of each biquaternion is the scalar of the other.*

Next, to express $\vec{B}$, $\vec{E}$ in terms of $\vec{D}$, $\vec{H}$, we can write

$$p_1 = r_1 + it_1 = \gamma_1 + i\delta_1.$$

where we have

$$\left.\begin{aligned} r_1 = 1 + \vec{D}\,; t_1 \;\; = \; -Q + \vec{H} = -G + \vec{H} \\ \gamma_1 = 1 + i\vec{H}\,; \delta_1 = -iQ + \vec{D} = -iG + \vec{D} \end{aligned}\right\} \tag{3, 5}$$

Putting further

$$\frac{Kt_1}{Tp_1} = a_1\,; \frac{K\delta_1}{Tp_1} = \beta_1$$

we can deduce, just as in III,

$$\left.\begin{aligned} \tfrac{1}{2}\{a_1, r_1\} = -\vec{B} \\ \tfrac{1}{2}\{\beta_1, r_1\} = -\vec{E} \end{aligned}\right\}. \tag{3, 6}$$

Also,

$$\left.\begin{aligned} \vec{S} = \tfrac{1}{2}[a_1, r_1] = \tfrac{1}{2} i\,[\beta_1, r_1] \\ (U + 1)^2 = r_1 Kr_1 (1 + a_1 Ka_1) \\ (V + 1)^2 = -\gamma_1 K\gamma_1 (1 + \beta_1 K\beta_1) \end{aligned}\right\}. \tag{3, 7}$$

We now proceed to biquaternions alternative to (2, 3) – (2, 4) and set up

$$p_2 = \sqrt{1 - 2\,\mathrm{M}} + \vec{\mathrm{B}} + \vec{\mathrm{D}} + \vec{\mathrm{S}} \qquad (3, 8)$$

$$p_3 = \sqrt{1 + 2\,\mathrm{N}} + i\vec{\mathrm{E}} + i\vec{\mathrm{H}} + \vec{\mathrm{S}} \qquad (3, 9)$$

where (as in I)

$$\mathrm{M} = (\vec{\mathrm{B}}\,\vec{\mathrm{D}})\,;\ \mathrm{N} = (\vec{\mathrm{E}}\,\vec{\mathrm{H}})\,;\ \vec{\mathrm{S}} = (\vec{\mathrm{D}} \times \vec{\mathrm{B}}) = (\vec{\mathrm{E}} \times \vec{\mathrm{H}}).$$

It will be seen that p_2 and p_3 are expressed only in terms of the primary vectors $(\vec{\mathrm{B}},\ \vec{\mathrm{D}})$ and $(\vec{\mathrm{E}},\ \vec{\mathrm{H}})$ respectively.

Also, $\mathrm{T}^2 p_2 = 1 + \vec{\mathrm{B}}^2 + \vec{\mathrm{D}}^2 + \vec{\mathrm{S}}^2$, since $(\vec{\mathrm{B}}\ \vec{\mathrm{S}}) = (\vec{\mathrm{D}}\ \vec{\mathrm{S}}) = 0$,

$$\left.\begin{aligned} i.e.,\ \mathrm{T}p_2 &= \mathrm{U} + 1 \\ \text{Similarly } \mathrm{T}p_3 &= \mathrm{V} + 1 \end{aligned}\right\} \qquad (3, 10)$$

To express the secondary vectors in terms of the primary ones, let us take (3, 8) first, and consider the quaternion

$$r_2 = \sigma + \vec{\mathrm{B}}, \qquad (\sigma = \sqrt{1 - 2\,\mathrm{M}})$$

and the bi-vector

$$t_2 = \vec{\mathrm{D}} + \vec{\mathrm{S}}, \text{ so that } q_2 = r_2 + t_2.$$

Then $r_2\ \mathrm{K}t_2 = -(\sigma + \vec{\mathrm{B}})(\vec{\mathrm{D}} + \vec{\mathrm{S}}) = -u\vec{\mathrm{E}} - (\sigma - 1)(\vec{\mathrm{D}} + \vec{\mathrm{S}}) + \mathrm{M}$,
by using the relation

$$u\vec{\mathrm{E}} = \vec{\mathrm{D}} + (\vec{\mathrm{B}} \times \vec{\mathrm{S}}). \qquad \text{[I, p. 581, eq. (31)]}$$

Similarly,

$$\mathrm{K}t_2 .\ r_2 = -(\vec{\mathrm{D}} + \vec{\mathrm{S}})(\sigma + \vec{\mathrm{B}})$$
$$= u\vec{\mathrm{E}} - (\sigma + 1)(\vec{\mathrm{D}} + \vec{\mathrm{S}}) + \mathrm{M}.$$

Hence,

$$(\sigma + 1)\, r_2\, \mathrm{K}t_2 (\sigma - 1)\, \mathrm{K}t_2\, r_2 = -2\,\sigma\, u\vec{\mathrm{E}} + 2\,\mathrm{M}$$
$$\text{or, } \sigma\,(r_2\,\mathrm{K}t_2 - \mathrm{K}t_2\, r_2) + (r_2\,\mathrm{K}t_2 + \mathrm{K}t_2\, r_2) = -2\,\sigma\, u\vec{\mathrm{E}} + 2\,\mathrm{M}$$
$$i.e., -u\vec{\mathrm{E}} = \tfrac{1}{2}\{r_2, \mathrm{K}t_2\} + \frac{1}{2\sigma}[r_2, \mathrm{K}t_2] - \frac{\mathrm{M}}{\sigma} \qquad (3, 11)$$

Again, let $\gamma_2 = \sigma + \vec{\mathrm{D}}\,;\ \delta_2 = \vec{\mathrm{B}} + \vec{\mathrm{S}}$; we have

$$\gamma_2 .\ \mathrm{K}\delta_2 = -(\sigma + 1)(\vec{\mathrm{B}} + \vec{\mathrm{S}}) + u\vec{\mathrm{H}} + \mathrm{M}$$

by using, $u\vec{\mathrm{H}} = \vec{\mathrm{B}} - (\vec{\mathrm{D}} \times \vec{\mathrm{S}})$.

Similarly,

$$\mathrm{K}\,\delta_2\gamma_2 = -(\sigma - 1)\,(\vec{\mathrm{B}} \times \vec{\mathrm{S}}) - u\vec{\mathrm{H}} + \mathrm{M}.$$

Hence,

$$(\sigma - 1)\ \gamma_2 \mathrm{K}\delta_2 - (\sigma + 1)\ \mathrm{K}\,\delta_2\gamma_2 = 2\,\sigma\, u\,\vec{\mathrm{H}} - 2\,\mathrm{M}$$

i.e.,
$$u\,\vec{\mathrm{H}} = \frac{1}{2\sigma}\{\gamma_2, \mathrm{K}\delta_2\} - \tfrac{1}{2}[\gamma_2, \mathrm{K}\delta_2] + \frac{\mathrm{M}}{\sigma} \qquad (3, 12)$$

(3, 11) and (3, 12) correspond to (2, 5) and (3, 6), but are not so elegant.

In a similar manner we can set up relations for the biquaternion p_3 by suitably choosing r_3, i_3, γ_3, δ_3.

4. *Generalised Action-Functions.*

Let us first consider Born's action-function in the form given by Infeld as the sum of a Lagrangian and Hamiltonian. In the case where $\mathrm{G} \neq 0$ we can write this in the form [see II, § 9, equation (9, 6)]

$$\mathrm{T} = \sqrt{2\phi + 2\psi + 4} \qquad (4, 1)$$

(but for a numerical term), where ϕ and ψ are the complex invariants given by

$$\left.\begin{aligned} \phi &= \mathrm{F} + i\mathrm{G} \\ \psi &= \mathrm{P} - i\mathrm{Q} \end{aligned}\right\} \qquad (4, 2)$$

with
$$\mathrm{F} = \tfrac{1}{2}\,(\vec{\mathrm{B}}^2 - \vec{\mathrm{E}}^2)\,;\ \mathrm{P} = \tfrac{1}{2}\,(\vec{\mathrm{D}}^2 - \vec{\mathrm{H}}^2).$$

We can also introduce the complex invariant

$$\mathrm{P} = \mathrm{R} + i\mathrm{S} \qquad (4, 3)$$

where R and S are given by

$$\mathrm{R} = \tfrac{1}{2}\{(\vec{\mathrm{B}}\,\vec{\mathrm{H}}) - (\vec{\mathrm{D}}\,\vec{\mathrm{E}})\}$$

$$\mathrm{S} = \tfrac{1}{2}\{(\vec{\mathrm{B}}\,\vec{\mathrm{D}}) + (\vec{\mathrm{E}}\,\vec{\mathrm{H}})\}.$$

The relations between the invariants [II, (2, 18) — (2, 19)] can be written in the form

$$\phi\,\psi = -\rho^2. \qquad (4, 4)$$

We proceed to show that it is possible to set up a biquaternion whose tensor is the action-function (4, 1).

Let
$$q = a + \vec{\mathrm{B}} - i\vec{\mathrm{E}} + \vec{\mathrm{D}} + i\vec{\mathrm{H}} \qquad (4, 5)$$

the bi-vector on the right-hand side of (4, 5) being the sum of the bi-vectors of p and p_1 whose tensors are the Lagrangian and Hamiltonian respectively. We have

$$\begin{aligned} \mathrm{T}^2 q &= a^2 + 2\phi + 2\psi + 2\,(\mathrm{S} + i\mathrm{R}), \text{ using } \mathrm{G} = \mathrm{Q} \\ &= a^2 + 2\phi + 2\,\psi + 2\,i\rho^* \end{aligned} \qquad (4, 6)$$

the (*) denoting the complex conjugate. If (4, 6) should be the square of (4, 1) we should have

$$a^2 + 2i\rho^* = 4$$

$$i.e.,\ a = 2\sqrt{1 - \tfrac{1}{2}i\rho^*},\ \text{and the biquaternion}$$

$$q = 2\sqrt{1 - \tfrac{1}{2}i\rho^*} + \vec{B} - i\vec{E} + \vec{D} + i\vec{H} \qquad (4, 7)$$

has its tensor equal to (4, 1). It is easy to see that the complex-conjugate biquaternion q^*, given by

$$q^* = 2\sqrt{1 + \tfrac{1}{2}i\rho} + \vec{B} + i\vec{E} + \vec{D} - i\vec{H} \qquad (4, 8)$$

has also the same tensor as q.

Let us now consider in the general case of action-functions an interpretation of the parameters λ and μ (II, § 6). It is well known that any quaternion could be reduced to a quotient of two vectors, and similarly a biquaternion can be taken as the quotient of two bi-vectors. Consider the quotient bi-quaternion

$$q' = \frac{\vec{B} - i\vec{E}}{\vec{D} + i\vec{H}} \qquad (4, 9)$$

and let the quaternion q be defined as the logarithm of q', *i.e.*,

$$q = \log q' \qquad (4, 10)$$

then, from the definition of the logarithm of a quaternion, we have

$$\left.\begin{aligned} Sq &= \log Tq' \\ \text{and,}\quad Vq &= \log Uq' \end{aligned}\right\}, \qquad (4, 11)$$

where S and V denote the scalar and vector, and T and U the tensor and versor of the quaternion. From (4, 9)

$$Tq' = \frac{T(\vec{B} - i\vec{E})}{T(\vec{D} + i\vec{H})} = \frac{\phi^*}{\psi^*}$$

$$= e^{\lambda - i\mu}. \qquad [\text{II, eq. (7, 1)}]$$

Hence $$Sq = \lambda + i\mu. \qquad (4, 12)$$

Thus the *complex parameter* $\lambda + i\mu$ *has been expressed as the scalar of a biquaternion.*

There does not appear to exist any simple interpretation of the vector of q.

EULERIAN PARAMETERS AND LORENTZ TRANSFORMATIONS.

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§ 1. *Introduction.*

In the theory of top motion there are three sets of quantities used, *viz.*, the Eulerian angles, the Cayley-Klein parameters, and the Eulerian parameters.[1] In a recent paper Sommerfeld[2] has considered an extension of the Cayley-Klein parameters to the problem of four-dimensional rotations, and brought out the connection of these generalised parameters to the Spinor calculus arising from Dirac's relativistic theory of the electron. I have considered in this paper a similar procedure for the Eulerian parameters, and shown that in this case the generalisation to four dimensions is more natural and that similar relationships to the Spinor calculus also exist.

§ 2. *Eulerian Parameters in Three Dimensions.*

These parameters denoted by ξ, η, ζ, Ω specifying an orthogonal transformation from (x, y, z) to (x', y', z') are four *real* numbers satisfying the condition

$$\xi^2 + \eta^2 + \zeta^2 + \Omega^2 = 1 \tag{1}$$

and the matrix of the transformation is given by

$$\begin{pmatrix} \xi^2 - \eta^2 - \zeta^2 + \Omega^2, & 2(\xi\eta + \zeta\Omega), & 2(\xi\zeta - \eta\Omega) \\ 2(\xi\eta - \zeta\Omega), & -\xi^2 + \eta^2 - \zeta^2 + \Omega^2, & 2(\eta\zeta + \xi\Omega) \\ 2(\xi\zeta + \eta\Omega), & 2(\eta\zeta - \xi\Omega), & -\xi^2 - \eta^2 + \zeta^2 + \Omega^2 \end{pmatrix} \tag{2}$$

If $(\xi'', \eta'', \zeta'', \Omega'')$ be the parameters corresponding to the resultant of two successive displacements $(\xi, \eta, \zeta, \Omega)$ and $(\xi', \eta', \zeta', \Omega')$, then

$$\Omega'' + \xi''\mathrm{i} + \eta''\mathrm{j} + \zeta''\mathrm{k} = (\Omega + \xi\mathrm{i} + \eta\mathrm{j} + \zeta\mathrm{k})(\Omega' + \xi'\mathrm{i} + \eta'\mathrm{j} + \zeta'\mathrm{k}), \tag{3}$$

where i, j, k are the usual quaternion operators, or if S, S′ and S″ be the corresponding quaternions, S″ = SS′, where

$$\mathrm{S} = \mathrm{i}\xi + \mathrm{j}\eta + \mathrm{k}\zeta + \Omega \qquad 3\,(a)$$

[1] See for *e.g.*, Whittaker, *Analytical Dynamics.*

[2] *Sitz. Wien. Akad.*, 1936, **Bd. 145.**

Also the vector $R = ix + jy + kz$ when subject to the orthogonal transformation $S = i\xi + j\eta + k\zeta + \Omega$ becomes R' given by

$$R' = SRS^{-1} \tag{4}$$

with $S^{-1} = -i\xi - j\eta - k\zeta + \Omega$, as can be verified from (2).

Also the Eulerian parameters and the Cayley-Klein parameters α, β, γ, δ are connected by the relations

$$\left.\begin{aligned} \xi &= \frac{1}{2}(\beta - \gamma); \quad \zeta = \frac{1}{2i}(\alpha - \delta) \\ \eta &= \frac{1}{2i}(\beta + \gamma); \quad \Omega = \frac{1}{2}(\alpha + \delta) \end{aligned}\right\} \tag{5}$$

Equation (1) then leading to

$$\alpha\delta - \beta\gamma = 1.$$

§ 3. *The Eulerian Parameters of the Lorentz-Transformation.*

In making a transition from three-dimensional transformations to rotations in space-time or Lorentz-transformations, the advantage in using Eulerian parameters lies in the fact that we can make this generalisation very simply by assuming ξ, η, ζ, Ω to be *complex* numbers instead of *real* numbers. The general Lorentz-transformation in space-time can then be specified by means of the eight quantities $A', B', \cdots D', A'' \cdots D''$ given by

$$\left.\begin{aligned} \xi &= A' + iA''; \quad \zeta = C' + iC'' \\ \eta &= B' + iB''; \quad \Omega = D' + iD'' \end{aligned}\right\} \tag{6}$$

Assuming the relation (1) of the previous section to hold this gives, equating real and imaginary parts,

$$\left.\begin{aligned} A'^2 + A''^2 + \cdots + D'^2 + D''^2 &= 1 \\ A'A'' + \cdots + D'D'' &= 0 \end{aligned}\right\} \tag{7}$$

We can also postulate that formula (4), which now becomes not a a quaternion formula but a biquaternion one, expresses the most general Lorentz-transformation in space-time provided suitable modifications are made in the representation of the operator S as given by (3 *a*). This last which is expressed by means of the quaternion units 1, i, j, k has to be represented by eight biquaternion units. This modification is most simply expressed by saying that we could still use the representation (3 *a*) for the operator S with the values for ξ, η, ζ, Ω in (6) provided we treat i, not as the complex number $\sqrt{-1}$ but as an operator γ

$$\gamma = \begin{pmatrix} 0, & -i \\ i, & 0 \end{pmatrix} \tag{8}$$

commuting with i, j, k.

Replacing i, j, k and $i\gamma$, $j\gamma$, $k\gamma$ by γ_{23}, γ_{31}, γ_{12} and γ_{14}, γ_{24}, γ_{34} respectively we easily verify the relations

$$\left.\begin{aligned} \gamma^2 = 1, \gamma_{ik}{}^2 = -1 \\ \gamma_{23}\gamma_{14} = \gamma_{31}\gamma_{24} = \gamma_{12}\gamma_{34} = \cdots = -\gamma \end{aligned}\right\} \tag{9}$$

writing the operator S as

$$\left.\begin{aligned} S = \gamma_{23}A' + \gamma_{31}B' + \gamma_{12}C' + D' + \gamma_{14}A'' + \gamma_{24}B'' + \gamma_{34}C'' + \gamma D'' \\ \text{and} \\ S^{-1} = -\gamma_{23}A' - \gamma_{31}B' - \gamma_{12}C' + D' - \gamma_{14}A'' - \gamma_{24}B'' - \gamma_{34}C'' + \gamma D'' \end{aligned}\right\} \tag{10}$$

we easily show, in virtue of (7) and (9), that

$$SS^{-1} = 1.$$

That the above formula correctly gives (4) as the general Lorentz-transformation can be verified by writing

$$\left.\begin{aligned} R = \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3 + \gamma_4 x_4 \\ R' = \gamma_1 x_1' + \gamma_2 x_2' + \gamma_3 x_3' + \gamma_4 x_4' \end{aligned}\right\}, \tag{11}$$

where (x_1, x_2, x_3, x_4) denote (x_1, y_1, z_1, ict) and $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ denote Dirac-matrices satisfying

$$\left.\begin{aligned} \gamma_{ik} + \gamma_{ki} = 2\delta_{ik} \\ \text{with } \gamma_{ik} = \gamma_i \gamma_k \end{aligned}\right\} \tag{12}$$

and observing that it is only necessary to assume that the γ's in (9) are the same as those in (12).

§ 4. *Relations to the Spinor-Calculus.*

Introducing the spinors (u^*, v^*) and (u, v) the stars denoting complex conjugates satisfying the unimodular transformations

$$\left.\begin{aligned} u' = (\Omega + i\zeta)\, u + (\xi + i\eta)\, v \\ v' = (-\xi + i\eta)\, u + (\Omega - i\zeta)\, v \end{aligned}\right\} \tag{13}$$

and

$$\left.\begin{aligned} u'^* = (\Omega - i\zeta)\, u^* + (\xi - i\eta)\, v^* \\ v'^* = (-\xi - i\eta)\, u^* + (\Omega + i\zeta)\, v^* \end{aligned}\right\} \tag{14}$$

the transformation scheme for the general Lorentz-transformation, which can be made to correspond to two mutually complex-conjugate unimodular transformations of two variables, can be written[3] as

$$\left.\begin{aligned} x_1 + ix_2 &\longleftrightarrow u^*v \\ x_1 - ix_2 &\longleftrightarrow v^*u \\ x_3 - ix_4 &\longleftrightarrow u^*u \\ -x_3 - ix_4 &\longleftrightarrow v^*v \end{aligned}\right\} \text{ or } \left.\begin{aligned} x_1 &\longleftrightarrow u^*v + v^*u \\ x_2 &\longleftrightarrow -i\,(u^*v - v^*u) \\ x_3 &\longleftrightarrow u^*u - v^*v \\ x_4 &\longleftrightarrow i\,(u^*u + v^*v) \end{aligned}\right\} \tag{15}$$

[3] See Kramers, *Quanten theorie des Electrons u der Strahlung*, 1937, 275. (*Hd.-u-Jahrbuch d. Chem. Phys.*, **Bd. 1, II**).

and this can be shown equivalent to (4). The expressions (13) and (14) for the spinor transformations in Eulerian parameters at once enables the writing of the relations $u^{\dagger} = v^{*}$ and $v^{\dagger} = -u^{*}$ for the spin-conjugate spinors in terms of the conjugate-complex ones.

If $u = u_1 + iu_2$, $v = v_1 + iv_2$, $u' = u_1' + iu_2'$, $v' = v_1 + iv_2'$ and putting the values $\Omega + i\zeta$, etc., in terms of A′, A″, etc., we can equate real and imaginary parts on the two sides of (13) and (14) and obtain transformations for the real and imaginary components of u, v in terms of the quantities A′, A″, · · · etc.

My thanks are due to Prof. B. S. Madhava Rao for suggesting the problem and kind guidance.

THEOREMS ON THE FUNCTIONAL LIMITS OF DERIVATIVES OF A FUNCTION AT INFINITY.

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§ 1. *Introduction.*

Hardy and Littlewood have proved the following theorem*:—

"If $f(x)$ is $O(\phi)$, and $f^n(x) = O(\psi)$ where ϕ and ψ are non-decreasing positive functions, then for $0 \leqslant r \leqslant n$

$$f^r(x) = O\left(\phi^{1-\frac{r}{n}} \cdot \psi^{\frac{r}{n}}\right).$$

In particular, if f and f^n are bounded so is f^r, for $0 \leqslant r \leqslant n$."

The object of this paper is, to go deeper into the question when f is bounded and f^n is bounded at least on one side, and examine the extent of the oscillation of $f^r(x)$ $(0 < r < n)$ at ∞. Let O_r be the oscillation of $f^r(x)$ at ∞. Then we prove the following theorems:—

(1) $$O_r \leqslant K_{r,n} \cdot O_0^{1-\frac{r}{n}} \cdot O_n^{\frac{r}{n}}$$

where $K_{r,n}$ are universal constants independent of $f, f', \cdots f^n$, but are functions of r and n only, and O_n = oscillation of $f^n(x)$ at ∞, and O_0 = oscillation of $f(x)$ at ∞, O_0 and O_n being finite $(0 < r < n)$.

(2) $$O_r \leqslant \overline{K}_{r,n} O_0^{1-\frac{r}{n}} \cdot (k_n)^{\frac{r}{n}} \qquad 0 < r < n,$$

where $\overline{K}_{r,n}$ are (as above) universal constants independent of $f, f', \cdots$ etc., and O_0 is as above, and $\underline{f_n(x)} = -k_n$ as $x \to \infty$ $(k_n > 0$ and finite).

A similar theorem when $\overline{f_n(x)} = k_n$ as $x \to \infty$ is given, follows easily; also in the interesting case when $k_n = 0$ it can be easily seen that the above inequality will be still true.

(3) We deduce certain consequences like Theorems I (a) and II (a). II (a) is more general than a theorem of Hardy and Littlewood.†

(4) We shew that the constants $K_{r,n}$ occurring in (1) are the best possible when $n = 2$, and when $n = 3$.

(5) If $0 < \alpha < 1$, and $f^\alpha(x)$ be the α-th fractional derivative of f, then $$O_\alpha \leqslant K_\alpha \cdot O_0^{1-\alpha} O_1^{\alpha}$$

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* *Proc. Lond. Math. Soc.*, 11 (New Series), 1913, p. 422.

† *Ibid.*, p. 426.

where O_0 = oscillation of $f(x)$ at ∞, O_1 = oscillation of $f'(x)$ at ∞ and K_a is a universal constant independent of f and f' and is a function of a only.

§ 2. *Theorem I.*

Let O_0 be the oscillation of $f(x)$ at ∞ and $(-k_2'$ and $k_2)$ the functional limits of $f''(x)$ at ∞ $(k_2 > 0,\ k_2' > 0)$. Let A_1 and $-B_1$ be the upper and lower functional limits of $f'(x)$ at ∞. We prove

$$\left(\frac{1}{k_2} + \frac{1}{k_2'}\right) \cdot \frac{A_1^2}{2} \leqslant O_0$$

$$\left(\frac{1}{k_2} + \frac{1}{k_2'}\right) \cdot \frac{B_1^2}{2} \leqslant O_0$$

and $$O_1 \leqslant \sqrt{2O_0 O_2}.$$

Let X be a sufficiently large number such that for all $x \geqslant X$

$$(1)\ -k_2' - \epsilon \leqslant f''(x) \leqslant k_2 + \epsilon,\ (2)|\, f(x_1) - f(x_2)\,| \leqslant O_0 + \epsilon$$

for $x_1, x_2 \geqslant X$.

Let x_0 be a number such that

$$f'(x_0) = A_1 + \epsilon' \ (\epsilon' \text{ being small})$$

Then for $h > 0$ $\quad f'(x_0 + h) \geqslant f'(x_0) - (k_2' + \epsilon)\, h$

and $\quad f'(x_0 - h) \geqslant f'(x_0) - (k_2 + \epsilon)\, h.$

[*Note.*—We take x_0 great enough so that $x_0 - \dfrac{A_1 + \epsilon'}{k_2' + \epsilon} \geqslant X$]

Let

$$h_1 = \frac{A_1 + \epsilon'}{k_2' + \epsilon}, \quad h_2 = \frac{A_1 + \epsilon'}{k_2 + \epsilon}$$

then in the interval, $x_0 - h_2 \leqslant x \leqslant x_0 + h_1$, the curve $y = f'(x)$ will be above the broken line POP_1 indicated in Fig. 1.

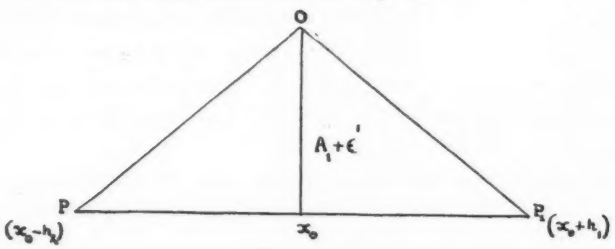


FIG. 1.

Therefore,

$$O_0 + \epsilon \geqslant f(x_0 + h_1) - f(x_0 - h_2) = \int_{-h_2}^{h_1} f'(x_0 + t)\, dt \geqslant \text{area of } \Delta\ POP_1$$

$$= \frac{(A_1 + \epsilon')^2}{2} \cdot \left\{\frac{1}{k_2 + \epsilon} + \frac{1}{k_2' + \epsilon}\right\}.$$

Since ϵ and ϵ' are arbitrarily small, we therefore have

$$\left.\begin{aligned} O_0 &\geqslant \tfrac{1}{2}\left(\frac{1}{k_2}+\frac{1}{k_2'}\right)A_1^2 \\ \text{Similarly,}\quad O_0 &\geqslant \tfrac{1}{2}\left(\frac{1}{k_2}+\frac{1}{k_2'}\right)B_1^2 \end{aligned}\right\} \qquad (1)$$

Further,

$$2\cdot\sqrt{\frac{2k_2k_2'}{k_2+k_2'}}\cdot O_0 \geqslant A_1 + B_1 = O_1.$$

But $$\frac{k_2+k_2'}{2} \geqslant \frac{2k_2k_2'}{k_2+k_2'}$$

Hence, $$2\sqrt{O_0\frac{(k_2+k_2')}{2}} = \sqrt{2\cdot O_0O_2} \geqslant O_1. \qquad (2)$$

We will now show by an example that (2) cannot be improved (*i.e.*, the value $\sqrt{2}$ for K_{12} cannot be replaced by anything smaller).

If non-existence of $f''(x)$ at an enumerable set be allowed then equality in (2) can actually occur.

Consider the function $\phi_0(t)$ defined as follows ; $(A>0,\ k>0)$,

$$\begin{aligned} \phi_0(t) &= \quad Kt \quad \text{in } \left(0, \frac{A}{K}\right) \\ &= \quad A - Kt \quad \text{in } \left(\frac{A}{K}, \frac{3A}{K}\right) \\ &= -A + Kt \quad \text{in } \left(\frac{3A}{K}, \frac{4A}{K}\right) \end{aligned}$$

and $\phi_0\left(t+\frac{4A}{K}\right) = \phi_0(t)$ at all other points.

In order to remove the discontinuity of $\phi_0'(t)$ at $P_1\left(\frac{A}{K}\right)$ and $P_3\left(\frac{3A}{K}\right)$, and at corresponding points $P_5, P_7, \cdots$, etc., draw a circle to touch OP_1 and P_1P_2 just below the point P_1 (Fig. 2). We can draw it so that P_1P_1' and

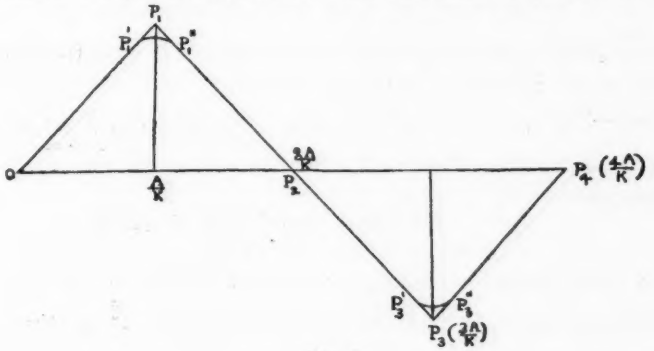


FIG. 2.

P_1P_1'' are very small (P_1' and P_1'' being the points of contact of the circle with OP_1 and P_1P_2). Also draw equal circles near P_3, P_5, etc., to touch the lines P_3P_1 and P_3P_4 at corresponding points $P_3'P_3''$, etc. (*i.e.* $P_3P_3' = P_3P_3'' = P_1P_1' = P_1P_1''$, etc.).

Let the function corresponding to the curve $OP_1'P_1''P_3'P_3''P_5'P_5''$, etc. be $\phi(t)$. Then it is easy to see that (i) $\phi(t)$ is periodic in $\frac{4A}{K}$, and (ii) $\int_0^{\frac{4A}{K}} \phi(t)\, dt = 0$, (iii) since the derivative on arc $P_1'P_1''$ lies between K and $-K$, $\phi'(t)$ lies between $-K$ and K.

Let $f(t) = \int_0^t \phi(t)\, dt$. Then $f(t)$ is periodic in $\frac{4A}{K}$ because of (ii) and maximum value of $f(t) = \int_0^{\frac{2A}{K}} \phi(t)\, dt$ = area of $\Delta OP_1P_2 - \epsilon$, where ϵ is very small.

$$= \frac{A^2}{K} - \epsilon \qquad (\text{in } 0 \leqslant x \leqslant \infty).$$

Minimum value of $f(t) = 0 \qquad (\text{in } 0 \leqslant x \leqslant \infty)$.

Therefore,

$$O_0 = \frac{A^2}{K} - \epsilon = \frac{A^2}{K}(1 - \epsilon_1).$$

It is easy to see that $O_1 = 2A - \epsilon' = 2A(1 - \epsilon_2)$

and $O_2 = 2K$.

Now,

$$\frac{O_1^2}{O_0O_2} = \frac{4A^2(1-\epsilon_2)^2}{\frac{A^2}{K}(1-\epsilon_1).\,2K} = 2(1-\epsilon_3)$$

since the ϵ's are arbitrarily small we therefore see that the value $\sqrt{2}$ for K_{12} cannot be replaced by anything smaller.

Note.—If $f''(x)$ is bounded on one side only, *i.e.*, $\overline{\underset{x\to\infty}{f''(x)}} = k_2$, in our procedure for proving Theorem I we make use of the interval $(x_0 - h_2, x_0)$ only, and obtain

$$A_1 \leqslant \sqrt{2O_1k_2} \text{ and, similarly } B_1 \leqslant \sqrt{2O_1k_2}$$

and, $$A_1 + B_1 = O_1 \leqslant 2.\sqrt{2}\ \sqrt{O_1k_2}.$$

We can easily devise an example somewhat similar to the one above to show that the above result is the best possible, *i.e.*, $2^{\frac{3}{2}}$ is the best value for $\overline{K}_{12}$.

Cor. I.—Assuming Hardy and Littlewood's theorem that if f and f^n are bounded so is f^r for $0 \leqslant r \leqslant n$, we get by repeated use of (2)

$$\frac{O_0}{O_1} \geqslant \frac{1}{2}\cdot\frac{O_1}{O_2} \geqslant \frac{1}{2^2}\frac{O_2}{O_3} \cdot \cdot \cdot \geqslant \frac{1}{2^r}\frac{O_r}{O_{r+1}} \cdot \cdot \cdot \geqslant \frac{1}{2^{n-1}}\frac{O_{n-1}}{O_n}.$$

Let $d_r = \dfrac{O_r}{O_{r+1}}$; the above becomes $d_0 \geqslant \dfrac{d_1}{2} \cdot \cdot \cdot \geqslant \dfrac{d_r}{2^r} \cdot \cdot \cdot \geqslant \dfrac{d_{n-1}}{2^{n-1}}$.

Then $d_0\, d_1 \cdot \cdot \cdot d_{r-1} = \dfrac{O_0}{O_r}$ and $d_r\, d_{r+1} \cdot \cdot\, d_{n-1} = \dfrac{O_r}{O_n}$.

Then,

$$\frac{O_0}{O_r} \geqslant \frac{(d_{r-1})^r}{2^{\frac{(r-1)r}{2}}}$$

and

$$\frac{O_r}{O_n} \leqslant 2^{\frac{(n-r)(n-r+1)}{2}} \cdot (d_{r-1})^{n-r}$$

$$\frac{O_0{}^{n-r}\, O_n{}^r}{O_r{}^n} = \left(\frac{O_0}{O_r}\right)^{n-r} \Big/ \left(\frac{O_r}{O_n}\right)^r \geqslant \frac{1}{2^{\frac{r(n-r)\cdot n}{2}}}.$$

Hence, $O_r \leqslant 2^{\frac{r(n-r)}{2}}\, O_0{}^{1-\frac{r}{n}} \cdot\, O_n{}^{\frac{r}{n}}$.

The values for K_{rn} obtained here are of orders ranging between

$$e^{a\cdot n} \text{ and } e^{a'\cdot n^2}.$$

They are too high values for K_{rn}; even for $n = 3$ they give as will be obvious from what follows too high values.

We will presently prove two theorems in which we can obtain values for K_{rn}, much smaller than the ones given by this corollary.

§ 3. *A Theorem of Hardy on Cesàro Summability.*

We will here shew that a theorem in Cesàro summability due to Hardy can be easily deduced from Theorem I.

Let $\phi_0(x)$ be such that $\left| x\,\dfrac{d\phi_0}{dx} \right| \leqslant k$.

Let $\phi_r(x) = \dfrac{r}{x^r}\int_0^x \phi_0(t)\,(x-t)^{r-1}\,dt$ (r being integral positive)

and let us define D by $D = x\dfrac{d}{dx}$.

Then $$D\,\phi_r = r\,(\phi_{r-1} - \phi_r)$$

and $$D^2\phi_r = r\,(D\phi_{r-1} - D\phi_r).$$

It is easy to prove

$$\text{(i)}\quad \mathrm{D}\phi_r = \frac{r}{x^r}\int_0^x \mathrm{D}\phi_0 \cdot (x-t)^{r-1}\,dt$$

$$\text{(ii)}\quad r\,(\mathrm{D}\phi_{r-1} - \mathrm{D}\phi_r) = \frac{r\,(r-1)}{x^2}\cdot \int_0^x (\mathrm{D}\phi_0 - \mathrm{D}\phi_1)\, t\,(x-t)^{r-2}\,dt.$$

Since $\mathrm{D}\phi_1 = \dfrac{1}{x}\displaystyle\int_0^x \mathrm{D}\phi_0\,dx$ and since $|\mathrm{D}\phi_0| \leqslant k$

$$|\mathrm{D}\phi_0 - \mathrm{D}\phi_1| \leqslant 2k.$$

By (ii) $$|\mathrm{D}^2\phi_r| \leqslant 2k. \qquad (r \geqslant 1).$$

Let O_r be oscillation $\phi_r(x)$ at ∞. Then by (1) in Theorem I, we have

$$\text{upper limit of } |\mathrm{D}\phi_r| \leqslant \sqrt{2kO_r}.$$

Hence if ϕ_r converges, $O_r = 0$

i.e., $$|\mathrm{D}\phi_r| \to 0$$

i.e., $$\phi_{r-1} - \phi_r \to 0 \qquad \text{as } x \to \infty$$

i.e., ϕ_{r-1} converges to a finite limit.

It is easy now to deduce that $\phi_0(x)$ converges to a definite limit if ϕ_r converges to a definite limit.

§ 4. *Theorem II.*

Let oscillation of $f(x)$ at ∞ be O_0 (finite) and lower limit of $f^n(x) = -k_n$ at ∞ $(k_n > 0)$ and $(k_n$ finite).

If A_{n-1} and $-B_{n-1}$ are the upper and lower functional limits of $f^{n-1}(x)$ at ∞ then

$$A_{n-1} \leqslant \frac{n}{2^n}\cdot O_0^{\frac{1}{n}} \cdot k_n^{1-\frac{1}{n}}$$

$$B_{n-1} \leqslant \frac{n}{2^n}\, O_0^{\frac{1}{n}} \cdot k_n^{1-\frac{1}{n}}.$$

Let $\Delta_{h_1} = f(x+h_1) - f(x)$ and $\Delta^2_{h_1,h_2} = \Delta_{h_2}\{f(x+h_1) - f(x)\}$, etc.

Then $$\Delta^{n-1}_{h_1 h_2 h_{n-1}} = \int_0^{h_1}\int^{h_2}\cdots\int_0^{h_{n-1}} f^{n-1}(x+u_1+u_2+\cdots+u_{n-1})\cdot$$

$$du_1, du_2 \cdots du_{n-1}$$

As in Theorem I choose x_0 sufficiently large such that

$$f^{n-1}(x_0) = A_{n-1} + \epsilon' \quad \text{and } f^n(x) \geqslant -(k_n + \epsilon)$$

(assuming $\Sigma h_r \leqslant$ a constant ; all the $h_r < 0$).

Then $$f^{n-1}(x_0 + \Sigma u_s) \geqslant (A_{n-1} + \epsilon') - (k_n + \epsilon) \cdot (\Sigma u_r).$$

Hence, $$\Delta_{h_1 \cdots h_{n-1}}^{n-1} \geqslant \int_0^{h_1} \cdots \int_0^{h_{n-1}} [A_{n-1} + \epsilon' - (k_n + \epsilon) \cdot (\Sigma u_r)]$$

$$du_1\, du_2 \cdots du_{n-1}.$$

The right-hand side

$$= h_1, h_2, h_{n-1} \left\{ A_{n-1} + \epsilon' - \frac{(k_n + \epsilon)}{2} (\Sigma h_r) \right\} = F(h_1, h_2, h_{n-1}).$$

It is easy to prove that F is a maximum when

$$h_1 = h_2 = \cdots = h_{n-1} = \frac{2 \cdot (A_{n-1} + \epsilon')}{n \cdot (k_n + \epsilon)} = h.$$

and

$$F(h, h, \cdots) = \left(\frac{2}{k_n + \epsilon}\right)^{n-1} \cdot \left(\frac{A_{n-1} + \epsilon'}{n}\right)^n.$$

Now $\Delta_h^{n-1} f = \Delta_h^{n-2} \cdot \{f(x + h) - f(x)\} \leqslant 2^{n-2}(O_0 + \epsilon'')$ [since x_0 is very large].

Therefore

$$2^{n-2}(O_0 + \epsilon'') \geqslant \left(\frac{2}{k_n + \epsilon}\right)^{n-1} \left(\frac{A_{n-1} + \epsilon'}{n}\right)^n$$

since the ϵ's are arbitrarily small, we have

$$O_0 \geqslant \frac{2 \cdot A^n{}_{n-1}}{n^n \cdot k_n{}^{n-1}}$$

i.e.,

$$A_{n-1} \leqslant \binom{n}{\frac{1}{2^n}} \cdot O_0^{\frac{1}{n}} k_n^{1-\frac{1}{n}}. \tag{3}$$

a similar inequality of B_{n-1} can be established.

Therefore, $O_{n-1} \leqslant 2^{1-\frac{1}{n}} \cdot n \cdot O_0^{\frac{1}{n}} \cdot k_n^{1-\frac{1}{n}}$, and step by step calculation leads to the inqeuality for O_r, and the value for K_{rn} is given in the note below.

In case O_n is finite it can be easily deduced from above that

$$O_{n-1} \leqslant n \cdot O_0^{\frac{1}{n}} \cdot O_n^{1-\frac{1}{n}}.$$

§ 5. *Generalization of a Theorem of Hardy-Littlewood.*

(i) In case f is bounded and $f^n(x)$ is bounded on one side (at least) then all the intermediate derivatives $f^r(x)$ are bounded. This is at once obvious from Theorem II ($0 \leqslant r < n$).

(ii) In case $O_0 = 0$, *i.e.*, f is convergent and f^n is bounded on one side at least then all the intermediate derivatives $f^r(x)$ converge to zero as $x \to \infty$ $(0 \leqslant r < n)$. This is also obvious from Theorem II.

(iii) In case O_0 is finite and either lower functional limit of $f^n(x) = 0$ or upper functional limit of $f^n(x) = O$ at ∞
then also, all the intermediate derivatives $f^r(x)$ $(0 < r < n)$ tend to zero. This needs a slight argument to prove it.

Suppose the lower function limit of $f^n(x) = 0$ at ∞;
then for all large x, $f^n(x) \geqslant -\epsilon$ (ϵ being small) and, we have as in Theorem II,

$$A_{n-1} \leqslant 2^{1-\frac{1}{n}} \cdot n\, O_0^{\frac{1}{n}} \cdot \epsilon^{1-\frac{1}{n}}$$

and $$B_{n-1} \leqslant 2^{1-\frac{1}{n}} \cdot n\, O_0^{\frac{1}{n}} \cdot \epsilon^{1-\frac{1}{n}}.$$

Since ϵ is arbitrarily small $A_{n-1} = B_{n-1} = 0$,
and now we can deduce easily $A_{n-2} = B_{n-2} = 0 \cdots$, etc.
Therefore the theorem follows. Similar argument when upper functional limit of $f^n(x) = 0$ is sufficient to prove the theorem.

This theorem is, as already noticed in the introduction, more general than one of Hardy and Littlewood (*ibid.*, p. 423) and is proved under more general conditions than there.

Note.—From (3), we can, by easy calculation, shew that for $0 < r < n$

$$\operatorname{Log} O_r - \left(1 - \frac{r}{n}\right)\operatorname{Log} O_0 - \frac{r}{n} \cdot \operatorname{Log} O_n \leqslant \operatorname{Log} K_{rn}$$
$$= r\left\{\frac{\operatorname{Log} r + 1}{r} + \frac{\operatorname{Log} r + 2}{r+1} + \cdots \frac{\operatorname{Log} n}{n-1}\right\}.$$

We obtained in Corollary I to Theorem I, for $\operatorname{Log} K_{rn}$ the value

$$\frac{r.(n-r)}{2} \cdot \operatorname{Log} 2.$$

$$\frac{\operatorname{Log} K_{rn} \text{ (in Theorem II)}}{r\frac{(n-r)}{2} \cdot \operatorname{Log} 2} = K \text{ (a constant)} . \frac{\frac{\operatorname{Log} r + 1}{r} + \cdots + \cdots + \frac{\operatorname{Log} n}{n-1}}{n-r}$$
$$= \text{utmost of order} \left(\frac{\operatorname{Log}^2 n}{n}\right)$$

Therefore (3) gives us much finer values for K_{rn} than Corollary I of Theorem I.

§ 6. *Theorem III.*

If $f^n(x)$ is bounded both ways we can, by the device of the following theorem, get much better values for $K_{r,n}$ than in Theorem II, if $n \geqslant 3$.

Theorem.—Let O_0 *and* O_n *be the oscillations of* $f(x)$ *and* $f^n(x)$ *at* ∞ *(both being finite). If* O_{n-2} *be the oscillation of* $f^{n-2}(x)$ *at* ∞, *then*

$$O_{n-2} \leqslant \frac{n}{2 \cdot 3^{\frac{n-2}{n}}} \cdot O_0^{\frac{2}{n}} O_n^{\frac{n-2}{n}}$$

and

$$O_{n-1} \leqslant \frac{n^{\frac{1}{2}}}{3^{\frac{n-2}{2n}}} \cdot O_0^{\frac{1}{n}} \cdot O_n^{1-\frac{1}{n}}.$$

Let $\Delta_{h_1} = f(x + h_1) - f(x - h_1)$ and $\Delta^2_{h_1,h_2} = \Delta_{h_1}\{\Delta_{h_2}\}$, etc.

Then $\Delta^{n-2}_{h_1 h_2 h_{n-2}} = \int_{-h_1}^{h_1} \int_{-h_2}^{h_2} \cdots \int_{-h_{n-2}}^{h_{n-2}} f^{n-2}(x + u_1 + u_2 + u_{n-2})$.

$$du_1 \cdot \cdot \cdot du_{n-2}.$$

Let A_{n-2} and $-B_{n-2}$ be the upper and lower functional limits of $f^{n-2}(x)$ at ∞, and let k_n and $-k_n'$ . . . . . $f^n(x)$ at ∞. Then, there will be an infinity of values of x (as large as you like) for which $f^{n-2}(x) = A_{n-2} + \epsilon$ (where ϵ is arbitrarily small) and $f^{n-2}(x)$ will be a maximum at that point. (This will not necessarily happen when $A_{n-2} = 0 = -B_{n-2}$ in which case Theorem III is self-evident. So we may assume $O_{n-2} \neq 0$.) Let x_0 be such a value. Assuming Σh_r to be bounded, since for large values of x, $-(k_n' + \epsilon) \leqslant f^n(x) \leqslant (k_n + \epsilon)$, we have $f^{n-2}(x_0 + \Sigma u_r) \geqslant f^{n-2}(x_0) - \frac{(k_n' + \epsilon)}{\lfloor 2} (\Sigma u_r)^2 = (A_{n-2} + \epsilon') - \frac{(k' + \epsilon)}{\lfloor 2} (\Sigma u_r)^2$ (since x_0 is a maximum point, $f^{n-1}(x_0) = 0$).

Then

$$\Delta^{n-2}_{h_1 h_2 h_{n-2}} f(x_0) \geqslant \int_{-h_1}^{h_1} \cdots \int_{-h_{n-2}}^{h_{n-2}} \left[(A_{n-2} + \epsilon') - \frac{(k_n' + \epsilon)}{\lfloor 2} (\Sigma u_r)^2 \right] \cdot$$

$$du_1 \, du_2 \cdot \cdot \cdot du_{n-2}.$$

The right-hand side $= F(h_1, h_2, \cdot \cdot \cdot h_{n-2})$

$$= 2^{n-2} \, h_1 \, h_2 \cdot h_{n-2} \left\{ (A_{n-2} + \epsilon') - \frac{(k_n' + \epsilon)}{\lfloor 3} (\Sigma h_r^2) \right\}.$$

Now F is a maximum when $h_1 = h_2 = h_{n-2} = \sqrt{\frac{6(A_{n-2} + \epsilon')}{n(k_n' + \epsilon)}} = h.$

Therefore, $\Delta_h^{n-2} \geqslant 2^{n-2} h^{n-2} \left\{(A_{n-2} + \epsilon') - \frac{(k_n' + \epsilon)}{\lfloor 3} \cdot (n-2) h^2\right\} =$

$$= 2^{n-2} \left\{\frac{6 (A_{n-2} + \epsilon')}{n \cdot (k_n' + \epsilon)}\right\}^{\frac{n}{2}-1} \cdot 2 \cdot \left(\frac{A_{n-2} + \epsilon'}{n}\right)$$

$$= \frac{2^{\frac{3n}{2}-2} \cdot 3^{\frac{n}{2}-1}}{n^{\frac{n}{2}}} \cdot \frac{(A_{n-2} + \epsilon')^{\frac{n}{2}}}{(k_n' + \epsilon)^{\frac{n}{2}-1}}.$$

Now $\Delta_h^{n-2} f(x_0) = \Delta_h^{n-3} \cdot \{f(x_0 + h) - f(x_0 - h)\} \leqslant 2^{n-3} (O_0 + \epsilon'')$.

Hence, $$O_0 + \epsilon'' \geqslant \frac{2^{\frac{n}{2}+1} \cdot 3^{\frac{n}{2}-1}}{n^{\frac{n}{2}}} \cdot \frac{(A_{n-3} + \epsilon')^{\frac{n}{2}}}{(k_n' + \epsilon)^{\frac{n}{2}-1}}.$$

Since the ϵ's are arbitrarily small we have

$$O_0 \geqslant \frac{2^{\frac{n}{2}+1} \cdot 3^{\frac{n}{2}-1}}{n^{\frac{n}{2}}} \cdot \frac{A_{n-2}^{\frac{n}{2}}}{(k_n')^{\frac{n}{2}-1}}$$

i.e., $$A_{n-2} \leqslant \frac{n}{2^{1+\frac{2}{n}} \cdot 3^{\frac{n-2}{n}}} \cdot O_0^{\frac{2}{n}} (k_n')^{1-\frac{2}{n}}. \qquad (4, a)$$

Similarly, we get

$$B_{n-2} \leqslant \frac{n}{2^{1+\frac{2}{n}} \cdot 3^{\frac{n-2}{n}}} \cdot O_0^{\frac{2}{n}} \cdot (k_n)^{1-\frac{2}{n}}. \qquad (4, b)$$

Thus,

$$A_{n-2} + B_{n-2} = O_{n-2} \leqslant \frac{n}{2 \cdot 3^{\frac{n-2}{n}}} \cdot O_0^{\frac{2}{n}} \left\{\frac{k_n^{1-\frac{2}{n}} + k_n'^{1-\frac{2}{n}}}{2^{\frac{2}{n}}}\right\}$$

$$\leqslant \frac{n}{2 \cdot 3^{\frac{n-2}{n}}} \cdot O_0^{\frac{2}{n}} (k_n + k_n')^{1-\frac{2}{n}}.$$

$$\leqslant \frac{n}{2 \cdot 3^{\frac{n-2}{n}}} \cdot O_0^{\frac{2}{n}} \cdot O_n^{1-\frac{2}{n}}. \qquad (4, c)$$

(Now $\frac{k_n^{1-\frac{2}{n}} + k_{n'}^{1-\frac{2}{n}}}{2} \leqslant \left(\frac{k_n + k_{n'}}{2}\right)^{1-\frac{2}{n}} = \frac{O_n^{1-\frac{2}{n}}}{2^{1-\frac{2}{n}}}$. Hence the inequality.)

Since by Theorem I, $O_{n-1} \leqslant \sqrt{2O_{n-2}O_n}$, substituting for O_{n-1} in (4, *c*) we obtain $O_{n-1} \leqslant \left(n^2 \Big/ 3^{\frac{n-2}{2n}}\right) \cdot O_0^{\frac{1}{n}} \cdot O_n^{1-\frac{1}{n}}$. (5)

§ 7. *Constants for* n = 3.

In case $n = 3$, we have from (4, *c*) and (5)

$$O_1 \leqslant \frac{3^{\frac{2}{3}}}{2} O_0^{\frac{2}{3}} \cdot O_3^{\frac{1}{3}}; \quad O_2 \leqslant 3^{\frac{1}{3}} O_0^{\frac{1}{3}} O_3^{\frac{2}{3}}.$$

We will give an example to shew that the constants K_{13}, K_{23}, given here are the best possible. If non-existence of $f'''(x)$ at an enumerable number of points be allowed then equality can actually occur in the above inequalities. Let $\phi(t)$ be defined as in the example under Theorem I and let

$$\phi_1(t) = \phi\left(t + \frac{A}{K}\right).$$

Then $\phi_1(t)$ will correspond to the graph (Fig. 3).

$Q_1P_1'P_3'P_3''P_5'Q_5$ in $0 \leqslant t \leqslant \frac{4A}{K}$.

The arcs Q_1P_1', $P_3'P_3''$, $P_5'Q_5$ being circular just as in the example under Theorem I.

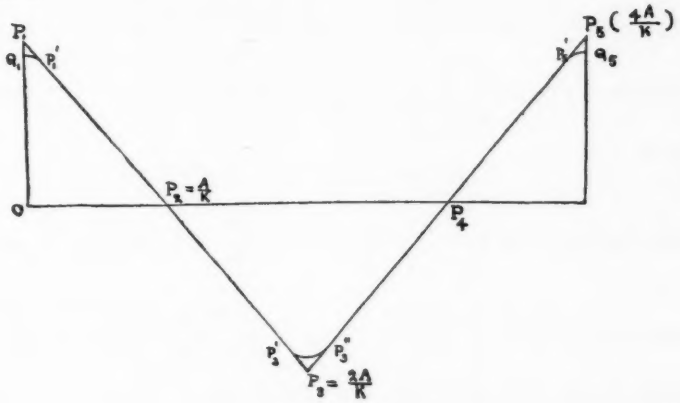


FIG. 3.

Then (i) $\phi_1(t)$ is periodic in $\frac{4A}{K}$ and (ii) $\int_0^{\frac{4A}{K}} \phi_1\, dt = 0$; (iii) oscillation of $\phi_1(t) = 2(A - \epsilon)$.

Let $\phi_2(t)$ be defined as $\int_0^t \phi_1 dt$. Then the graph of $\phi_2(t)$ will be slightly deformed parabolas as shown here (Fig. 4).

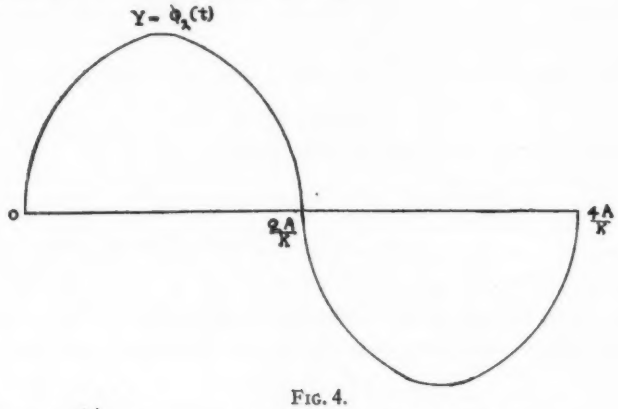


FIG. 4.

ϕ_2 is periodic in $\frac{4A}{K}$ because of (ii) and further as is obvious.

$$\phi_2\left(t + \frac{2A}{K}\right) = -\phi_2(t)\,; \quad \text{therefore} \int_0^{\frac{4A}{K}} \phi_2\, dt = 0.$$

Therefore the function $f(t) = \int_0^t \phi_2(t)\, dt$ is periodic in $\frac{4A}{K}$

and maximum of $f(t) = \int_0^{\frac{2A}{K}} \phi_2\, dt = \frac{2}{3}\frac{A^3}{K^2} - \epsilon'' = \frac{2}{3}\frac{A^3}{K^2}(1 - \epsilon_3)$; minimum of $f(t) = 0$.

Similarly,

$$\text{maximum of } f'(t) = \int_0^{\frac{A}{K}} \phi_1(t)\, dt = \frac{A^2}{2K} - \epsilon' = \frac{A^2}{2K}(1 - \epsilon_2)$$

$$\text{minimum of } f'(t) = -\int_0^{\frac{A}{K}} \phi_1(t)\, dt = -\left(\frac{A^2}{2K} - \epsilon'\right) = -\frac{A^2}{2K}(1 - \epsilon_2)$$

oscillation of $f''(t) = 2(A - \epsilon) = 2A(1 - \epsilon_1)$

oscillation of $f''(t) = 2K$.

Therefore,

$$O_0 = \frac{2}{3}\frac{A^3}{K^2}(1-\epsilon_3);\ O_1 = \frac{A^2}{K}(1-\epsilon_2);\quad O_2 = 2A(1-\epsilon_1) \text{ and } O_3 = 2K.$$

Now

$$\frac{O_1}{O_0^{\frac{2}{3}}\,O_0^{\frac{1}{3}}} = \frac{\frac{A^2}{K}(1-\epsilon_2)}{\left\{\frac{2}{3}\frac{A^3}{K^2}(1-\epsilon_3)\right\}^{\frac{2}{3}}(2K)^{\frac{1}{3}}} = \frac{3^{\frac{2}{3}}}{2}(1-\epsilon_4)$$

and,

$$\frac{O_2}{O_0^{\frac{1}{3}}\,O_2^{\frac{2}{3}}} = \frac{2A(1-\epsilon_1)}{\left\{\frac{2}{3}\frac{A^3}{K^2}(1-\epsilon_3)\right\}^{\frac{1}{3}}(2K)^{\frac{2}{3}}} = 3^{\frac{1}{3}}(1-\epsilon_5).$$

We therefore see that $K_{13} = \frac{3^{\frac{2}{3}}}{2}$ and $K_{23} = 3^{\frac{1}{3}}$ are the best possible values (when $n = 3$). We have shown the same to be true of K_{12} in § 2.

8. On Fractional Derivatives.

Let O_0 and O_1 be oscillations of $f(x)$ and $f'(x)$ at ∞

Let

$$f_0^{\alpha}(x) = \frac{1}{\lceil 1-\alpha}\cdot\frac{d}{dx}\int_0^x \frac{f(t)}{(x-t)^{\alpha}}\,dt \qquad (0 < \alpha < 1).$$

Right-hand side $=$

$$\frac{1}{\lceil 1-\alpha}\left\{\frac{f(0)}{x^{\alpha}} + \int_0^x \frac{f'(t)}{(x-t)^{\alpha}}\,dt\right\}.$$

Since we are interested in the asymptotic behaviour of $f^{\alpha}(x)$ at ∞ the first term may be omitted. We will next shew that from our point of view, it is matter of indifference what origin we take, for

$$f_a^{\alpha}(x) = \frac{1}{\lceil 1-\alpha}\frac{d}{dx}\int_a^x \frac{f(t)}{(x-t)^{\alpha}}\,dt = \frac{1}{\lceil 1-\alpha}\left\{\frac{f(a)}{(x-a)^{\alpha}} + \int_a^x \frac{f'(t)}{(x-t)^{\alpha}}\,dt\right\}.$$

As before the first term is of no importance and

$$\int_a^x \frac{f'(t)}{(x-t)^{\alpha}}\,dt - \int_a^x \frac{f'(t)}{(x-t)^{\alpha}}\,dt = \int_0^a \cdots = O\left(\frac{1}{x^{\alpha}}\right).$$

Hence we will consider $\frac{1}{\lceil 1-\alpha}\int_a^x \frac{f'(t)}{(x-t)^{\alpha}}\,dt$ as defining $f^{\alpha}(x)$.

Now

$$\lceil 1-\alpha \cdot f_a^{\alpha}(x+a) \sim \int_a^{x+a} = \int_0^x \frac{f'(t+a)}{(x-t)^{\alpha}}\,dt = \int_0^x \frac{f'(a+x-t)}{t^{\alpha}}\,dt.$$

$$= \int_0^{x_1} + \int_{x_1}^{x} = I_1 + I_2 \qquad (0 < x_1 < x).$$

$$I_1 = f'(a + x - \theta x_1)\ \frac{{x_1}^{1-a}}{1-a} \quad \text{and} \quad I_2 = \frac{1}{{x_1}^a} \int_{x_1}^{\xi} f'(a + x - t)\, dt.$$

Let A_1 and $-B_1$ be the upper and lower functional limits of $f'(x)$ at ∞. Taking 'a' fairly large we have

$$I_1 + I_2 \leqslant (A_1 + \epsilon)\ \frac{{x_1}^{1-a}}{1-a} + \frac{O_0 + \epsilon}{{x_1}^a} = F(x_1).$$

$F(x)$ is a minimum when $x = a \cdot \dfrac{O_0 + \epsilon}{A_1 + \epsilon}$.

Taking for x_1 the value $x_1 = a \cdot \dfrac{O_0 + \epsilon}{A_1 + \epsilon}$.

We get, $\sqrt{1-a}\, f^a(x) = I_1 + I_2 \leqslant$

$$\frac{(A_1 + \epsilon)}{1-a} \cdot \left(a \cdot \frac{O_0 + \epsilon}{A_1 + \epsilon}\right)^{1-a} + \frac{O_0 + \epsilon}{a^a} \left(\frac{A_1 + \epsilon}{O_0 + \epsilon}\right)^a \leqslant \frac{(A_1 + \epsilon)^a \cdot (O_0 + \epsilon)^{1-a}}{(1-a) \cdot a^a}$$

Hence if A_a and $-B_a$ are the functional limits of $f^a(x)$ at ∞ we have,

$$A_a \leqslant \frac{{O_0}^{1-a}\, {A_1}^a}{\sqrt{2-a}\, .\, a^a}.$$

$$B_a \leqslant \frac{{O_0}^{1-a}\, {B_1}^a}{\sqrt{2-a}\, a^a}.$$

Since

$${A_1}^a + {B_1}^a \leqslant 2 \cdot \left(\frac{A_1 + B_1}{2}\right)^a = 2^{1-a}\, (A_1 + B_1)^a$$

we have

$$O_a = A_a + B_a \leqslant \frac{2^{1-a}}{\sqrt{2-a} \cdot a^a} \cdot\ {O_0}^{1-a}\, {O_1}^a. \tag{6}$$

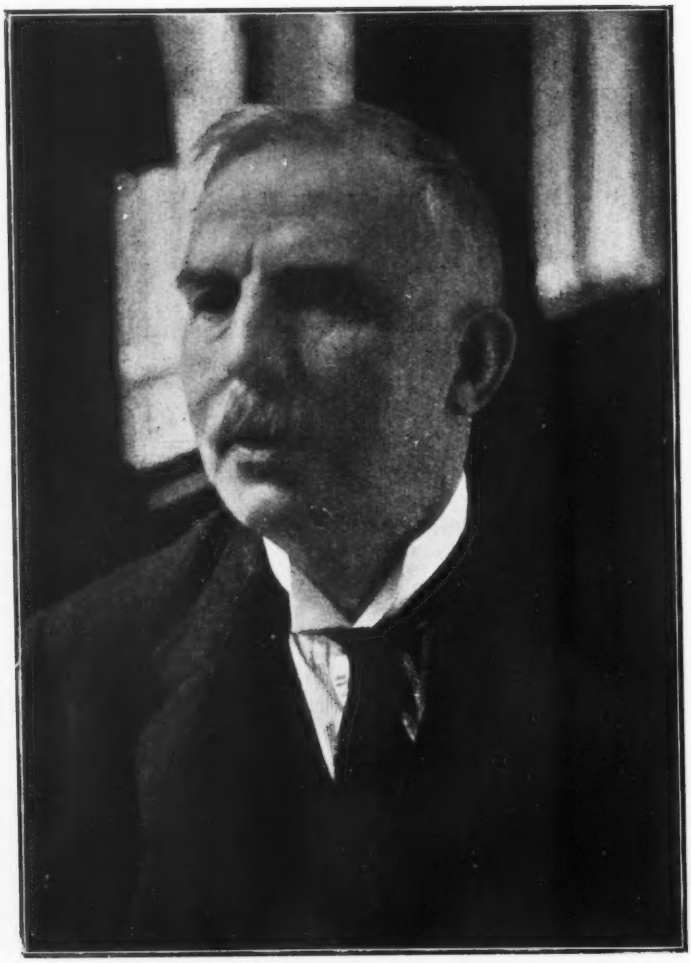
Since

$$\frac{2^{1-a}}{\sqrt{2-a} \cdot a^a} \leqslant 2e^{\frac{1}{e}} \quad \text{we have} \quad O_a \leqslant 2 \cdot e^{\frac{1}{e}} \cdot\ {O_0}^{1-a} \cdot\ {O_1}^a \cdot \qquad 0 < a < 1.$$

An immediate corollary is that when either f or f' converges and the other remains bounded, all the intermediate derivatives $f^a(x)$ tend to zero as $x \to \infty$.

Note.—We could equally well define by $f^a(x)$ by $\dfrac{1}{\sqrt{1-a}} \displaystyle\int_x^{\infty} \frac{f'(t)}{(x-a)^a}\, dt.$

(for large x), the latter integral being convergent since f is bounded and we get the same result as (6) for this definition of $f^a(x)$.



LORD RUTHERFORD OF NELSON

Born: 1871 *Died: 1937*

OBITUARY.

LORD RUTHERFORD OF NELSON.

LORD RUTHERFORD OF NELSON was not only the most distinguished Honorary Fellow of the Indian Academy of Sciences, but the most outstanding figure in the whole field of contemporary science. In company with the rest of the scientific world, the Academy mourns his loss, and it is therefore fitting that we should pay our tribute of respect to his memory.

Rutherford was born at Nelson, in New Zealand, in the year 1871. He received his early education there and at Christchurch College, Canterbury, N.Z. From the first he showed that experimental skill and aptitude for research which were to become his outstanding qualities. In 1895, at the age of 24, he went to England and joined the Cavendish Laboratory as a research student under J. J. Thompson. It thus happened that Rutherford entered the scientific world at a crucial time. The great achievements of classical physics had reached their climax, and a new era was beginning. Rutherford became one of a brilliant band of young physicists who were quick to see the significance of the new discoveries. From the first he took the lead, and became, in the truest sense of the term, a pioneer, leading the way into the unexplored territories of atomic physics.

By 1898 Rutherford had already so far distinguished himself that he was appointed Macdonald professor of Physics in McGill University, Montreal, where he remained until 1907. During these years he unravelled, in a remarkable series of researches, the complicated phenomena of radioactive transformation. In collaboration with Soddy he produced the transformation theory, now so familiar that we fail to appreciate its revolutionary character. Many distinguished physicists of the older school were slow to accept Rutherford's conclusions, for the principle of the permanence of the chemical elements seemed one of the most securely established results of nineteenth century science.

Neither do we appreciate to the full the experimental difficulties of those early days. As Rutherford himself frequently said in later years, the chief feature of more recent developments has been the astonishing improvement in our experimental resources and technique. Not possessing those resources, Rutherford nevertheless worked out the relations of the

various radioactive bodies, and the character of their radiations, with such power and completeness that his conclusions stand to-day.

From 1907 to 1919 Rutherford was Langworthy Professor of Physics in the University of Manchester. The earlier part of this period saw a long series of important researches, dealing chiefly with the properties of the α-particle. In 1911 came what has justly been called "the finest of all his great contributions to physics", the nuclear theory of the atom. Nor was this merely a speculation, but a necessary conclusion from his experiments on the scattering of α-particles. Again, looking back on this work of which the results are so familiar to-day, one cannot fail to be impressed both by Rutherford's intellectual power and his mastery of the experimental method.

During the War there was a necessary lull in pure research, and Rutherford gave much time to scientific work of national importance. In 1919 he was called to the Cavendish Chair of Experimental Physics, which he held until his death. Under his direction the Cavendish Laboratory has produced a continuous stream of research of the greatest value and importance. It is from 1919 that we may date the attack on the atomic nucleus, for it was then that Rutherford was able to demonstrate the disruption of the nitrogen nucleus by α-particles. The following years saw great developments of technique, and scattering experiments began to yield information about nuclear fields of force. In 1932 the discovery of the neutron by Chadwick initiated a new burst of discovery in which the Cavendish Laboratory took the leading part, and which has continued with unabated vigour to the present day.

For more than 40 years Rutherford has been the acknowledged leader in the field of radioactive research, and it is scarcely possible to exaggerate the value and importance of his work. He was gifted with profound physical insight, and with remarkable experimental skill. These gifts, combined with boundless energy and enthusiasm, made his great work possible. To no other worker of this generation do we owe so much.

Rutherford was great not only as a physicist, but as a man. He was very friendly and very human. In spite of the many calls upon his time and attention he made a point of knowing personally every worker in the laboratory, and took a personal interest in each investigation. He had a rich sense of humour, and it is impossible to forget that freshness and spontaneity which made any contact with him a delight. There was nothing affected or artificial about him, both in speech and manner he was simple and direct. Those who have at any time worked in his laboratory mourn his loss not only as a great teacher, but as a friend.

It can be said of Rutherford, as of very few, that by his work he changed the whole face of science. To find his equal we must go back to Faraday, perhaps to Newton. He belongs of right to that small but illustrious company of men who by their single-minded devotion to truth have been chiefly responsible for the advancement of human knowledge. At the height of his power he has passed from us, but his name will continue to be held in honoured remembrance, and his work will remain as an enduring memorial.

H. J. T.

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Page 296, para 1, line 11—

for " *l*-methyl cinnamate" *read* " *l*-menthyl-cinnamate".